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ROLE OF INTERNAL ROTATIONS IN SELECTED MAGNETIC FLUID APPLICATIONS

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Dissipation takes place in colloidal magnetic fluid having particles rotated out of equilibrium, and the fluid can support a state of asymmetric magnetic and viscous stress yielding a weakening of forces. This work examines the influence of shear rate on magnetization of magnetic fluid, and the heating of magnetic fluid subjected to oscillatory and rotating fields. The importance to devices and systems is noted, and the consistency of stress tensor and field energy descriptions is illustrated.

Introduction

Colloidal magnetic fluid (ferrofluid) in which the magnetic particles are oriented out of alignment with the direction of the local magnetic field is in a state of disequilibrium that can exert strong influence on behavior of systems. A related response occurs in time-varying magnetic field that is uniaxial. This work examines aspects of three different systems exhibiting internal rotations that can exert substantial effects in ferrofluid devices and systems.

Demagnetization in shear flow

The initial topic bears on the performance of magnetic fluid rotary seals, magnetic fluid bearings, and related systems. When a magnetized colloidal magnetic fluid is subjected to viscous shear the magnetic particles of the composite tend to be rotated out of alignment with the applied magnetic field.

The following develops an analysis of the phenomenon in its greatest simplicity, in reference to the sketch of Fig. 1 representing plane parallel shear flow. Conditions in this system are closely related to the problem of viscosity increase in the presence of steady magnetic field. The analysis is based on the equation of conservation of internal angular momentum for a magnetic fluid, which can be written as follows [1, 2]

$$\rho\frac{\mathrm{D}\mathbf{s}}{\mathrm{D}t} = \rho\mathbf{G} + \nabla\cdot\mathsf{C} + \mathbf{A}, \tag{1}$$

$\mathbf{s}$ is internal angular momentum per unit mass of magnetic fluid (brought on by the rotation of magnetic particles, or spin for short), $\mathbf{G}$ is magnetic body couple density per unit mass, C is surface couple stress tensor, and $\mathbf{A}$ is the vector of the antisymmetric part of the viscous stress tensor, i.e., $\mathbf{A} = \mathbf{vec}\mathsf{T}_\mathbf{a}$ representing the conversion rate of angular momentum between external and internal forms. Equation (1) is written in the Cauchy or unconstituted form.

In steady state shear motion s is constant, hence D**s**/Dt = 0. The $\nabla\cdot\mathsf{C}$ represents transport of angular momentum by the diffusion of spinning magnetic particles down a gradient of spin; numerical estimate indicates

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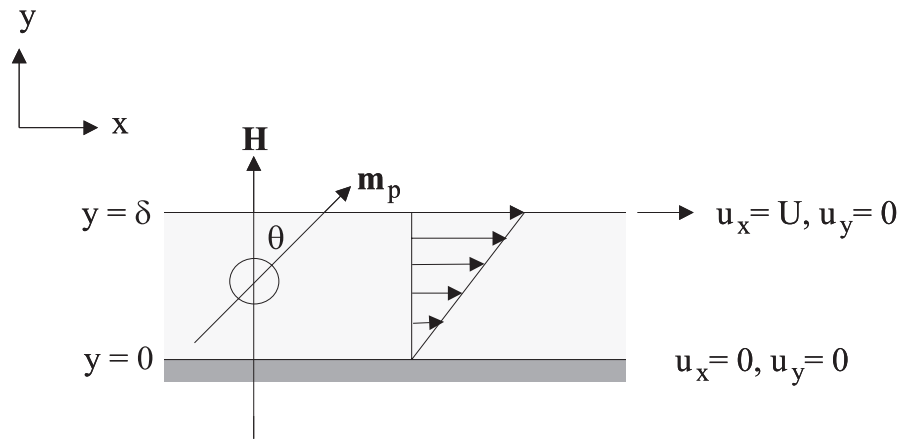

Fig. 1. Plane parallel shear flow indicating rotation of a particle (magnified).

it is a negligible influence [2, 3], hence it is neglected here. Relationships for $\rho\mathbf{G}$ and $\mathbf{A}$ are as follows[2]:

$$\rho\mathbf{G} = \mu_0\mathbf{M} \times \mathbf{A}, \tag{2}$$

$$\mathbf{A} = 4\zeta(\boldsymbol{\omega}_\mathrm{f} - \boldsymbol{\omega}_\mathrm{p}), \tag{3}$$

where μ_0 is permeability of free space, $\mathbf{M} = \mu_0^{-1}\mathbf{B} - \mathbf{H}$ is magnetization of the magnetic fluid, ζ is termed the vortex viscosity, $\boldsymbol{\omega}_\mathrm{f} = \boldsymbol{\Omega}$ is fluid rotational rate, $\boldsymbol{\Omega} = 1/2\nabla \times \mathbf{v}$ is the vorticity, $\mathbf{v}$ the vector velocity, and $\boldsymbol{\omega}_\mathrm{p}$ the angular spin rate of particles; $\boldsymbol{\omega}_\mathrm{p}$ is proportional to $\mathbf{s}$. Equation (3) for $\mathbf{A}$ is a constitutive relationship. Equation (1) now reduces to

$$0 = \mu_0\mathbf{M} \times \mathbf{H} + 4\zeta(\boldsymbol{\omega}_\mathrm{f} - \boldsymbol{\omega}_\mathrm{p}). \tag{4}$$

This relationship states that the magnetic couple plus the viscous couple exerted on unit volume of magnetic fluid sum to zero. The viscous couple represents the drag due to rotation rate of bulk fluid relative to the rotation rate of particles. Because (4) contains $\mathbf{M}$ and $\boldsymbol{\omega}_\mathrm{p}$ as unknowns, an additional equation is required to make the system determinate. The relaxation equation of Shliomis [4] is useful for this purpose.

$$\frac{\mathbf{D}\mathbf{M}}{\mathbf{D}t} = \boldsymbol{\omega}_\mathrm{p} \times \mathbf{M} - \frac{1}{\tau}(\mathbf{M} - \mathbf{M}_0), \tag{5}$$

where $\mathbf{M}_0$ is the equilibrium magnetization in field $\mathbf{H}$ and τ is a relaxation time constant. Equation (4) can be

[2]The magnetic body couple density is not a constitutive relationship, although it is sometimes regarded as such. Given the magnetic stress tensor T valid for un-equilibrated magnetic fluid, the relationship follows as a corollary. Thus, the surface torque density of stress vector $\mathbf{t}_n$ acting at a point on the surface located distance $\mathbf{r}$ from an arbitrary origin is given by $\mathbf{r} \times \mathbf{t}_n$. Because $\mathbf{t}_n = \mathbf{n} \cdot \mathsf{T}$, where $\mathbf{n}$ is the unit normal to the surface, the density can be expressed as $\mathbf{r} \times (\mathbf{n} \cdot \mathsf{T}) = -(\mathbf{n} \cdot \mathsf{T}) \times \mathbf{r} = -\mathbf{n} \cdot (\mathsf{T} \times \mathbf{r})$. The integral of this term over the surface transforms by Gauss's theorem to integration over the volume enclosed by the surface of the integrand $-\nabla \cdot (\mathsf{T} \times \mathbf{r})$. By vector identity this term can be expanded as $\mathbf{r} \times (\nabla \cdot \mathsf{T}) + \mathbf{vec}\mathsf{T}$, where $\nabla \cdot \mathsf{T} = \mathbf{f}$ with $\mathbf{f}$ the force density. The first term tends to produce external or orbital rotation of fluid, and the second is body couple density, producing particle rotations that are internal to the fluid. From the expression for T of Eq. (31), $\mathbf{vec}\mathsf{T} = -\epsilon : \mathsf{T} = \mu_0\mathbf{M} \times \mathbf{H} = \rho\mathbf{G}$, as was to be proven. The torque per unit volume exerted on a whole body in response to an applied field $\mathbf{H}_0$ is given by $\mu_0\mathbf{M} \times \mathbf{H}_0$ which has the same form except $\mathbf{H}_0$ substitutes for $\mathbf{H}$. There is no way to know the correctness of the expression given by Eq. (2) *a priori*.

solved algebraically for $\boldsymbol{\omega}_{\mathrm{p}}$ giving

$$\boldsymbol{\omega}_{\mathrm{p}} = \frac{\mu_0 \mathbf{M} \times \mathbf{H}}{4\zeta} + \boldsymbol{\Omega}. \tag{6}$$

Eliminating $\boldsymbol{\omega}_{\mathrm{p}}$ between (6) and (5) gives, with slight rearrangement, for the steady state with $\mathrm{D}\mathbf{M}/\mathbf{D}t = 0$,

$$\boldsymbol{\Omega} \times \mathbf{M} = \frac{1}{\tau}(\mathbf{M} - \mathbf{M}_0) + \frac{\mu_0}{2\zeta}(\mathbf{M} \times \mathbf{H}) \times \mathbf{M}. \tag{7}$$

In this problem, $\boldsymbol{\Omega} = \Omega\mathbf{j}$, $\mathbf{H} = H\mathbf{k}$, $\mathbf{M}_0 = M_0\mathbf{k}$, and $\mathbf{M} = \mathbf{M}_0 + \mathbf{m}$, where $\mathbf{m} = m_x\mathbf{i} + m_y\mathbf{j} + m_z\mathbf{k}$ represents the deviation of magnetization from its equilibrium value. Substituting these expressions into (7) yields three algebraic equations, one for each coordinate direction. From the y-direction equation, $m_y = 0$, and the other two equations may be written as:

$$\left(1 + \frac{m_z}{M_0}\right)\left[\Omega\tau - P\left(\frac{m_1}{M_0}\right)\right] - \frac{m_1}{M_0} = 0, \tag{8}$$

$$\frac{m_z}{M_0} - \frac{m_1}{M_0}\left[P\left(\frac{m_1}{M_0}\right) - \Omega\tau\right] = 0, \tag{9}$$

where $P = \mu_0 M_0 H\tau/4\zeta$ is a dimensionless group representing a ratio of magnetic to viscous body couple. Eliminating m_z/M_0 yields the following cubic equation for $r = Pm_x/M_0\Omega\tau$:

$$r^3 - 2r^2 + r\left[1 + (P+1)/(\Omega\tau)^2\right] - P/(\Omega\tau)^2 = 0. \tag{10}$$

Numerics. For a typical magnetic fluid rotary seal, $\mu_0 M_0 = 0.06$ tesla, and $\mu_0 H_0 = 1.5$ tesla, and for the typical seal magnetic fluid, $\tau = 2 \times 10^{-5}$ seconds and ζ = 0.2 kg/ms. The corresponding value of P = 0.895. For simplicity in the numerical calculations P is set equal to 1. Equation (10) can then be expressed as follows:

$$\left(\frac{m_x}{M_0}\right)(\Omega\tau)^2 - \left[2\left(\frac{m_x}{M_0}\right)^2 + 1\right](\Omega\tau) + \left(\frac{m_x}{M_0}\right)^3 + 2\left(\frac{m_x}{M_0}\right) = 0. \tag{11}$$

For a fixed value of m_x/M_0, this is a quadratic equation for $\Omega\tau$. Assigning a series of numerical values to m_x/M_0 and calculating the corresponding values of the positive root leads to the plot of Fig. 2. At small values of shear ($\Omega\tau < 1$), m_x/M_0 increases with increase of $\Omega\tau$. A peak value of m_x/M_0 is reached beyond which m_x/M_0 steadily decreases. The initial increase clearly is due to the reorientation of particles in the shear field; the subsequent decrease shows the predominating influence of thermal disorientation.

Values of m_y/M_0, may now be calculated from corresponding values of m_x/M_0, using either (8) or (9). The component of magnetization along the direction of the applied field decreases monotonically with $\Omega\tau$ while the tilt angle θ of the magnetization vector relative to the applied field direction is given by

$$\theta = \arctan\frac{m_x/M_0}{1 + m_y/M_0}. \tag{12}$$

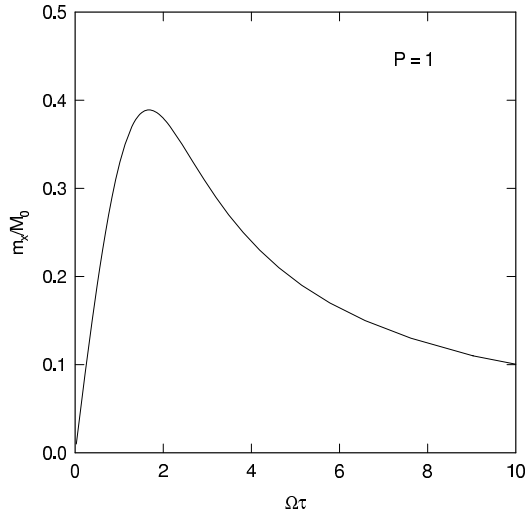

Fig. 2. Shear stress produces component of magnetization orthogonal to the magnetic field.

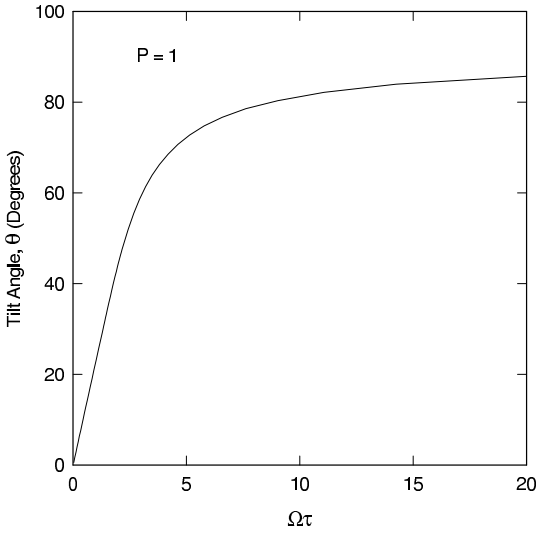

Fig. 3. Deviation angle of magnetization relative to the applied magnetic field.

Figure 3 shows the dependence of θ on shear rate; the asymptotic value of tilt angle is $\pi/2$.

The magnitude M of the magnetization is given by

$$\frac{M}{M_0} = \sqrt{\left(1 + \left(\frac{m_y}{M_0}\right)^2 + \left(\frac{m_x}{M_0}\right)^2\right)}. \tag{13}$$

M/M_0 is plotted in Fig. 4, where it can be seen that the shear not only tilts the magnetization but it also decreases its magnitude.

Influence on performance. The maximum pressure difference that can be sustained by one stage of a magnetic fluid seal is dictated by the linear momentum equation for un-equilibrated magnetic fluid, which can be expressed in the following form.

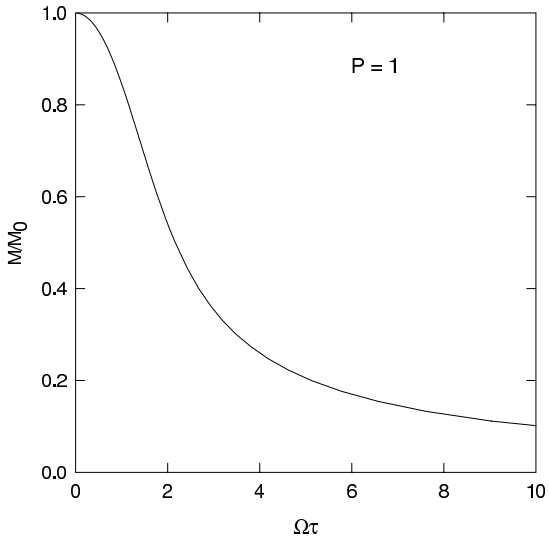

Fig. 4. Magnetization magnitude is reduced in shear flow.

$$\rho\frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} = -\nabla p + \rho\mathbf{g} + \eta\nabla^2\mathbf{v} + \mu_0\mathbf{M}\cdot\bigtriangledown\mathbf{H} + \frac{\mu_0}{2}\nabla\times(\mathbf{M}\times\mathbf{H}). \tag{14}$$

In the static seal, $\mathbf{M}$ is parallel to $\mathbf{H}$ and the last two terms on the right side collapse to $\mu_0 M\nabla H$, where $\mathbf{M}$ and $\mathbf{H}$ are magnitudes. $\mathrm{D}\mathbf{v}/\mathrm{D}t = 0$ and thus, for a horizontal system, the equation further reduces to $0 = -\nabla p + \mu_0 M\nabla H$, which yields upon integration $\triangle p = \mu_0 M\int_1^2 M\mathrm{d}H$, where the limits of integration refer to conditions at the opposite interfaces of magnetic fluid in a seal stage. In the case of the rotating seal, integration requires evaluation of the two vectorial magnetic terms on the right side of (14). The requisite values of $\mathbf{M}$ are determined from the prior analysis developed above. As shown previously, the magnitude of $\mathbf{M}$ decreases when the ferrofluid is sheared, and this reduces the magnitude of the magnetic terms. The numerical solution for pressure difference is evidently a soluble problem, albeit complex, and a solution is not attempted here. Suffice it to say, that the influence of shear on magnetization could be a limiting factor in the pressure holding integrity of magnetic fluid seals and similar devices.

Energy dissipation in alternating field

Magnetic fluid can be heated by exposure to alternating magnetic field. The phenomenon holds interest in medical hyperthermia for the treatment of cancer and other applications [5, 6]. A thorough analysis of the heating process appears not to be given previously. Investigators have published power law expressions giving a dependence on frequency and field magnitude, but the expressions have a limited range of validity that is not specified, and the influence of other parameters is not given.

The problem holds interest in the present context as the heating that occurs depends principally oftentimes on viscous fluid friction attendant to the rotation of the magnetic particles of the ferrofluid.

The differential $\mathrm{d}W$ of work done on the ferrofluid per unit volume in generating a differential change of magnetic field is given by [7]

$$\mathrm{d}W = \mathbf{H}\cdot\mathrm{d}\mathbf{B}. \tag{15}$$

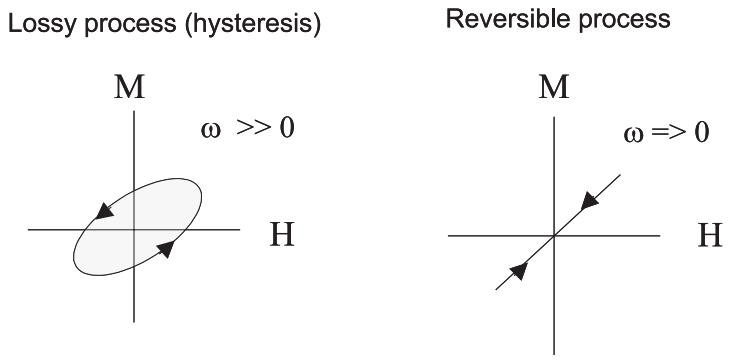

Fig. 5. Hysteresis curves for magnetic fluid subjected to oscillatory magnetic field.

When the applied field is polarized along a single axis, the vectors $\mathbf{H}$ and $\mathbf{B}$ are parallel at all times, hence the work expression reduces to $\mathrm{d}W = H\mathrm{d}B$, where H and B are magnitudes. Using the defining equation for magnetization, $B = \mu_0(H + M)$, the work done over a cycle can be expressed as

$$\oint \mathrm{d}W = \mu_0 \oint H\mathrm{d}M - \mu_0 \oint M\mathrm{d}H. \tag{16}$$

The diagram for a lossy process compared with a reversible process is illustrated in Fig. 5. The energy dissipated per cycle corresponds to the enclosed area of a cycle. Representing the applied field as the real part of $H(t) = H_0\exp(i\omega t)$, where H_0 is field amplitude, ω is cyclic frequency of the applied field, and t is time, a complex susceptibility $\chi = \chi(\omega)$ is defined as

$$\chi = \chi' - i\chi'', \tag{17}$$

where χ', χ'' are real and such that $M(t)$ is the real part of $\chi H(t)$. That is,

$$M(t) = \mathrm{Re}\left[\chi H(t)\right]. \tag{18}$$

Expressed in terms of reals this takes the following form:

$$M(t) = \left(\chi' \cos\omega t + \chi'' \sin\omega t\right). \tag{19}$$

The susceptibility components can be found using the Shliomis relaxation equation, given previously as Eq. (5). In a stationary medium subjected to oscillatory field, $\omega_\mathrm{p} = 0$ and $\mathbf{v} = 0$ so the relaxation equation reduces to

$$\frac{\partial M}{\partial t} = -\frac{1}{\tau}\left(M - M_0\right), \tag{20}$$

where, as previously, τ is a relaxation time, and

$$M_0 = M_0(t) = \chi_0 \mathrm{Re}\left[H(t)\right], \tag{21}$$

where M_0 is the equilibrium magnetization in the applied field whose value at a given instant of time is $H(t)$. The χ_0 is the initial value of susceptibility, $\chi_0 = (\partial M/\partial H)_0$.

In a ferrofluid that is not too concentrated the magnetization is given by the Langevin relationship $M/M_{\rm s} = \coth\xi - 1/\xi$, where $M_{\rm s}$ is the saturation value of M, $\xi = \mu_0 mH/kT$ with $m = M_{\rm d}V_{\rm p}$ the magnetic moment of a particle, $M_{\rm d}$ the domain magnetization of a particle, k the Boltzmann constant, and $V_{\rm p}$ the volume of a particle. The corresponding initial susceptibility χ_0 for steady field is given by $\chi_0 = \mu_0 M_{\rm d}^2 V_{\rm p}\phi/3kT$. In computations it will be convenient to use the chord susceptibility $\chi_{\rm c} = M/H$.

$$\frac{\chi_{\rm c}}{\chi_0} = \frac{3}{\xi}\left(\coth\xi - \frac{1}{\xi}\right). \tag{22}$$

Note that as $\xi \to 0$, $\chi_{\rm c} \to \chi_0$. Equation (20) now suffices to determine the frequency dependent susceptibility χ. Substituting for $M(t)$ from (18) and $M_0(t)$ from (21) yields

$$\chi' = \frac{\chi_{\rm c}}{1 + i\omega\tau} \tag{23}$$

and thus,

$$\chi' = \frac{\chi_{\rm c}}{1 + (\omega\tau)^2}, \qquad \chi'' = \frac{\omega\tau}{1 + (\omega\tau)^2}\chi_{\rm c}. \tag{24a,b}$$

The relationships of (24a,b) are identical to the Debye spectra of polar molecules in the absence of a constant field. Use of the chord susceptibility yields a lower bound on the susceptibility components, hence on the heating; in most cases χ_c will not deviate greatly from χ_0. The spectra for monodisperse particles are plotted in Fig. 6 which shows that the real and imaginary components have values that cross at the peak value of χ''. Also shown for comparison is the computed spectra for a polydispersion of magnetic particles assuming a log normal particle size distribution with standard deviation $\sigma = 0.1$. The curves are shifted greatly showing why measurements of actual magnetic fluids display variance from the monodisperse prediction.Cyclic power, which equals dissipation rate $\Phi = W \cdot f$, may now be formulated using the integration indicated in Eq. (16), substituting M from (19) with $H = {\rm Re}[H(t)] = H_0\cos\omega t$. The result obtained is $\Phi = \mu_0\pi\chi'' H_0^2 f$ in which there is no dependence on χ'. Substituting for χ'' from (24b) and with $f = \omega/2\pi$ yields

$$\Phi = \frac{\mu_0\chi_{\rm c}H^2}{2\tau}\frac{(\omega\tau)^2}{1 + (\omega\tau)^2} \tag{25}$$

with $\chi_{\rm c}$ given by (22). At low frequencies Φ is proportional to ω^2 and becomes independent of ω at high frequencies. The dependence on H is not simple quadratic as $\chi_{\rm c}$ decreases with increasing H.

To complete the analysis a relationship for τ is required. The two types of relaxation in magnetic fluids are Brownian (rotation of the particle with magnetic moment locked to the crystal axis) and Neel (rotation of magnetic moment with respect to the crystal axis):

$$\tau_{\rm B} = \frac{3\eta V_{\rm H}}{kT}, \qquad \tau_{\rm N} = \frac{\exp\Gamma}{f_0\Gamma^{1/2}}, \ \Gamma \geq 2, \tag{26a,b}$$

where η is carrier fluid viscosity, $V_{\rm H}$ hydrodynamic volume of a particle, and $\Gamma = KV_{\rm p}/kT$, where K is the anisotropy constant and $V_{\rm p}$ the magnetic volume of a particle. The $V_{\rm H}$ is estimated as the coated volume of a sterically stabilized particle. Relaxation takes places in parallel by both mechanisms with the effective time τ given by

$$\frac{1}{\tau} = \frac{1}{\tau_{\rm B}} + \frac{1}{\tau_{\rm N}}. \tag{27}$$

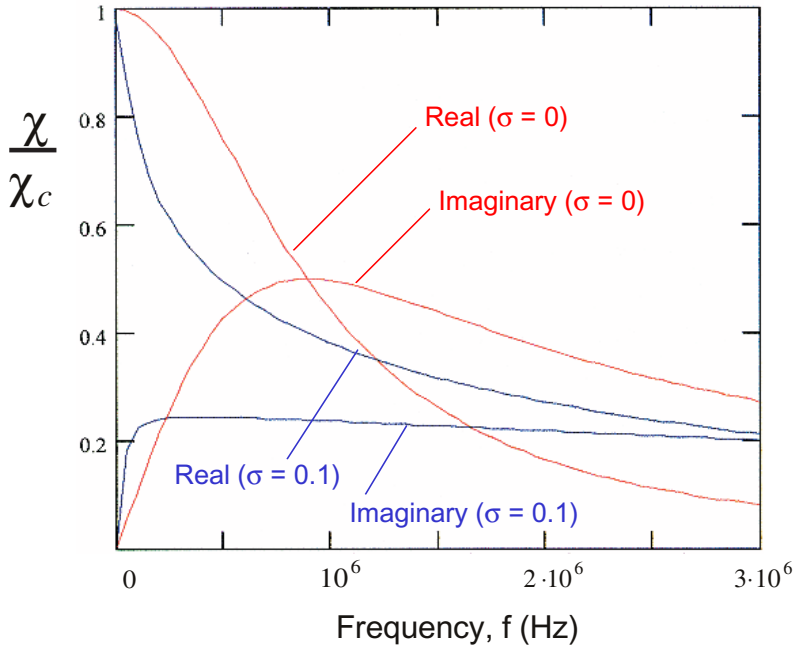

Fig. 6. Susceptibility spectra of magnetite particles. $\sigma = 0$ denotes monodisperse particles; $\sigma = 0.1$ for the polydispersion; mean particle diameter 8.9 nm; B_0 = 0.06 tesla.

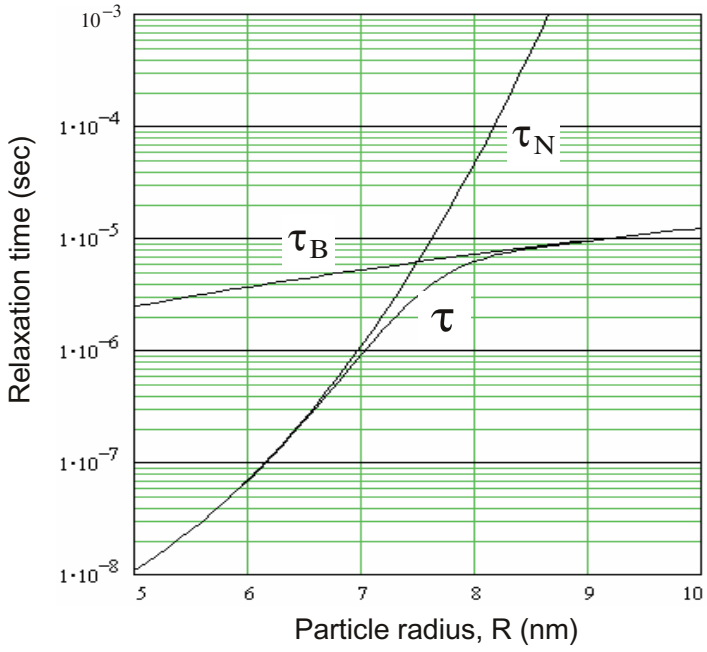

Fig. 7. Relaxation times for magnetite particles in tetradecane as function of particle size. Refer to text for physical properties.

Figure 7 illustrates the dependence of τ_B, τ_N, and τ on particle size for nanoparticles of magnetite in hydrocarbon carrier liquid (M_d = 446 kA/m, K = 23 kJ/m^3, η = 0.00235 kg/ms). The τ_N is much smaller than τ_B for small particles, but increases much more rapidly with increasing particle size and the crossover at particle radius of about 7.5 nm occurs above the 5 nm mean radius of particles in a typical preparation.

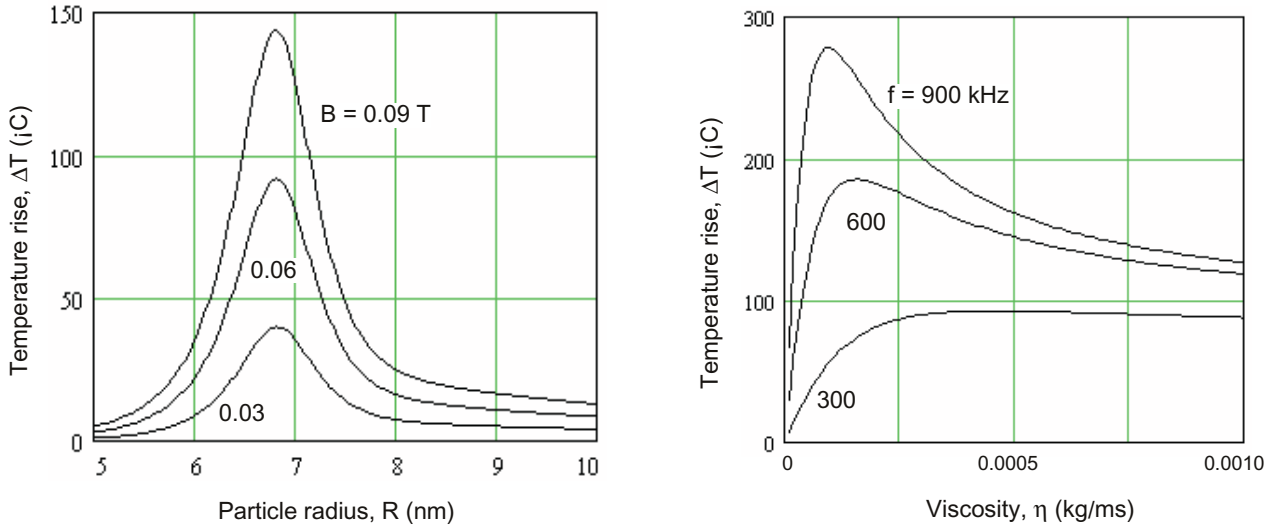

Fig. 8. Adiabatic temperature rise in monodisperse magnetite having particle volume fraction 0.10; heating duration 0.2 seconds.

Figure 8 presents calculated values of adiabatic temperature rise vs. the size of monodisperse magnetite particles. Peak heating coincidentally is obtained with particles near the nominal size, but for particles of this size the viscosities yielding peak heating rates are inaccessible with usual carrier liquids. In summary, the model presented here should be useful for defining optimal magnetic fluids and systems.

Response to rotating field

The spin-up motion of magnetic fluid in an applied rotating field has long held fascination for investigators. Many theories of the spin-up motion were advanced with the matter finally resolved, we think, by critical experiments and theory of Rosensweig *et al.* [3]. The coupling of field to the fluid is established via magnetic tangential surface stresses present in un-equilibrated magnetic fluid, with the direction of motion dependent on the curvature of the meniscus. With an upwardly concave meniscus, rotation of the fluid is counter to the rotational sense of the applied magnetic field; an upwardly convex meniscus produces rotation of fluid in the same direction as that of the field. Finally, with a flat meniscus the fluid is motionless.

Due to the no slip wall condition it follows there can be no motion of an enclosed volume of magnetic fluid filling its container. (The minute motion predicted by spin diffusion theory is neglected [8]). Here examination is made of a different aspect of the system – the work and dissipation associated with the rotation of the field relative to the fluid. The sketch of Fig. 9 illustrates an embodiment of such a system in which the field is produced by the rotation of a permanent magnet where it is easily understood that a torque must be applied to rotate the magnet; however, the results apply equally well to a system in which field is produced by electrical currents. An objective is to show that the thermodynamic formulation of dissipation agrees with analysis based on the Maxwell stress tensor of un-equilibrated magnetic fluid. Thus, a prescription for energy dissipated in one cycle of the applied rotary field is given by Eq. (16). Previously, this relationship was used to evaluate dissipation rate in linearly polarized field. As the relationship purports to hold general validity it should be applicable as well to the current problem. Referring to Fig. 10 for nomenclature, at a given instant of time the magnetic field $\mathbf{H}$ in the medium is oriented at an angle $\theta = \omega t$ with respect to the x-axis, and the magnetization lags behind by angle α which is constant in the steady state. From the geometry of the figure,

$$H_x = H_0 \cos \omega t, \quad H_y = H_0 \sin \omega t,$$
$$M_x = M_0 \cos(\omega t - \alpha), \quad M_y = M_0 \sin(\omega t - \alpha). \tag{28}$$

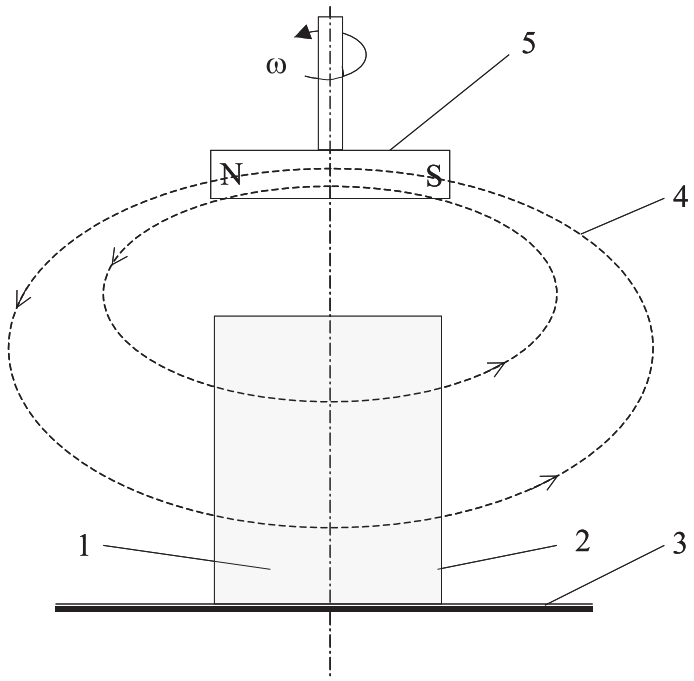

Fig. 9. Sketch of magnetic fluid *1* enclosed in filled container *2* attached to a stationary base *3* and subjected to rotating magnetic field *4* of magnet *5*.

Thus, the dissipated work per cycle is given by

$$W = -\mu_0 \oint \mathbf{M} \cdot \mathrm{d}\mathbf{H}$$

$$= -\mu_0 M_0 H_0 \omega \int_0^{2\pi/\omega} [\cos(\omega t - \alpha) \sin \omega t + \sin(\omega t - \alpha) \cos \omega t] \mathrm{d}t$$

$$= 2\pi \mu_0 M_0 H_0 \sin \alpha \tag{29}$$

and the energy dissipation rate per unit volume, Φ, is given by

$$\Phi = W \cdot f = \mu_0 \omega M_0 H_0 \sin \alpha, \tag{30}$$

where cyclic frequency $f = \omega/2\pi$.

Stress tensor method. Next it is desired to verify this result based on a different direction of attack employing the magnetic fluid stress tensor T. The form of T unencumbered with strictive terms and applicable to un-equilibrated magnetic fluid is [9, 10]

$$\mathsf{T} = -\frac{\mu_0}{2} H^2 \mathsf{I} + \mathbf{BH}. \tag{31}$$

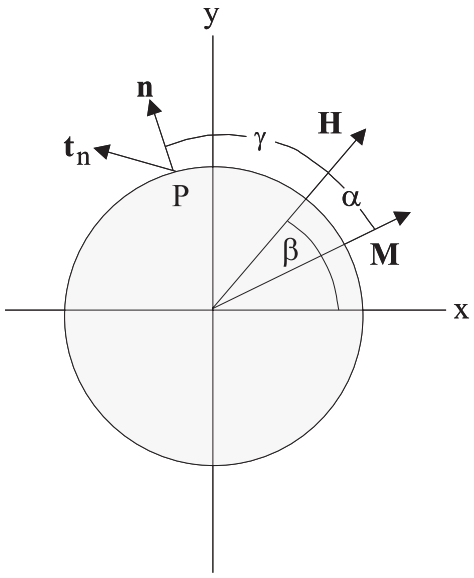

Fig. 10. Nomenclature sketch for system in rotating field.

The associated vector stress $\mathbf{t}_n$ acting upon the surface at a point P, see Fig. 10, having unit outward facing normal $\mathbf{n}$ is given by

$$\mathbf{t}_{\mathrm{n}} = \mathbf{n} \cdot \mathsf{T} = -\frac{\mu_0}{2} H^2 \mathbf{n} + B_{\mathrm{n}} \mathbf{H}. \tag{32}$$

The magnitude of the tangential component of the vector force is given by

$$\mathrm{t_t} = |(\mathsf{I} - \mathbf{nn}) \cdot \mathbf{t}_{\mathrm{n}}| = |B_{\mathrm{n}} H_{\mathrm{t}}|. \tag{33}$$

Referring to the nomenclature of Fig. 10, the components B_{n} and H_{n} are given by

$$B_{\mathrm{n}} = \mu_0 (M_{\mathrm{n}} + H_{\mathrm{n}}) = \mu_0 M_0 \cos(\gamma + \alpha) + \mu_0 H_0 \cos\gamma, \tag{34a}$$

$$H_{\mathrm{t}} = H_0 \sin\gamma. \tag{34b}$$

Accordingly, the torque acting on the whole surface is given by

$$Total\ torque = R \oint_S |B_{\mathrm{n}} H_{\mathrm{t}}| \mathrm{d}S. \tag{35}$$

Substituting for B_{n} and H_{t} from (34 a and b) with $\mathrm{d}S = 2\pi RL\mathrm{d}\theta$ and carrying out the integration from $\gamma = 0$ to 2π yields

$$Total\ torque = V \mu_0 M_0 H_0 \sin\alpha, \tag{36}$$

where $V = \pi R^2 L$ is system volume, R is radius of the cylindrical vessel and L its height. Dividing the expression by V yields the term $\mu_0 M_0 H_0 \sin\alpha$. The associated power input equals the dissipation rate Φ, that therefore is

found as the product of this term and ω giving

$$\Phi = \mu_0 \omega M_0 H_0 \sin\alpha, \tag{37}$$

which is identical to the dissipation rate obtained previously as Eq. (30). Thus, the field energy and stress tensor formulations of dissipation rate are compatible with each other.

The fluid mechanical framework for a medium supporting asymmetric stress furnishes a further perspective. The body couple density $\mathbf{A}$ of stress tensor T is given by $\mathbf{A} = \mathbf{vec}\mathsf{T} = -\epsilon : \mathsf{T}$, where ϵ is the polyadic alternator. With T specified by Eq. (31), $\mathbf{A} = \mu_0 \mathbf{M} \times \mathbf{H} = \mu_0 M_0 H_0 \sin\alpha$, yielding a total torque consistent with Eq. (36). When $\mathbf{M}$ is oriented other than parallel with field $\mathbf{H}$, body couple density is present and asymmetric stress is inherent in the system.

Components of the dissipation [11]**.** The total dissipation Φ due to the rotating field is given by Eq. (37). A portion $\Phi_{\rm c}$ of the dissipation is due to the coherent rotation of the suspended magnetic particles, and the remainder $\Phi_{\rm nc}$ is assumed due to noncoherent rotations causing relaxation of the magnetization.

$$\Phi = \Phi_{\rm c} + \Phi_{\rm nc}. \tag{38}$$

Relaxation proceeds with equal numbers of oppositely oriented rotations of the magnetic particles.

The volumetric density of body torque due to rotation of particles is given by the term $\mu_0 M_0 H_0 \sin\alpha$, as shown previously. Because particles rotate at angular rate $\omega_{\rm p} < \omega$, the corresponding dissipation rate is

$$\Phi_{\rm c} = \mu_0 \omega_{\rm p} M H \sin\alpha. \tag{39}$$

Thus,

$$\Phi_{\rm nc} = \Phi - \Phi_{\rm c} = \mu_0 (\omega - \omega_{\rm p}) M H \sin\alpha. \tag{40}$$

Using the relaxation relationship of (5) it can be shown [2] that

$$\tan\alpha = (\omega - \omega_{\rm p})\tau, \quad M = M_0 \cos\alpha. \tag{41a,b}$$

Eliminating $\omega_{\rm p}$ and M from (40) yields

$$\Phi_{\rm nc} = \frac{\mu_0 M_0 H}{\tau} \frac{\tan^2\alpha}{1 + \tan^2\alpha}. \tag{42}$$

$\Phi_{\rm nc} \approx 0$ when $\tan\alpha \approx 0$, and occurs, as can be shown from the cubic equation for $\tan\alpha$, see Eq. (8.68) in [2], in strong fields ($\xi \gg 1$) when $2\omega\tau/\xi \ll 1$ in which case for Brownian relaxation the inequality takes the form $H \gg 6\eta\omega/M_{\rm d}$ and is independent of particle size, and in weak fields ($\xi \ll 1$) when $\omega\tau/\xi \ll 1$, where $\xi = \mu_0 M_{\rm d} H V_{\rm p}/kT$ is the argument of the Langevin function. Maximum dissipation corresponds to $\tan\alpha \gg 0$ giving $\Phi_{\rm nc,max} = \mu_0 M_0 H/\tau$, and is reached in high intensity fields ($\xi \gg 1$) when $2\omega\tau/\xi \gg 1$, and in low intensity fields ($\xi \leq 1$) when $\omega\tau \gg 1$. As one result, $\Phi_{\rm nc}$ is small in low frequency fields and maximum at high frequencies.

Assuming constant susceptibility, $\chi = M_0/H$, then $H = M/\chi\cos\alpha$ from (41b) and from (40) and (41b) $\Phi_{\rm nc} = \mu_0(\omega - \omega_{\rm p})^2 M^2 \tau/\chi$. The term $M^2(\omega - \omega_{\rm p})^2 = (\mathbf{M} - \mathbf{M_0})^2/\tau^2$ as can be verified by substitution into

the relaxation Eq. (5). Thus, $\Phi_{\rm nc}$ can be expressed alternatively in a form that demonstrates its positive nature:

$$\Phi_{\rm nc} = \frac{\mu_0(\mathbf{M} - \mathbf{M_0})^2}{\chi\tau}. \tag{43}$$

This response in the spin-up flow system makes an interesting comparison with the response in purely oscillatory field where the dissipation is due entirely to the non-coherent mechanism of magnetic relaxation.

Conclusion

This work visited three systems of magnetic fluid interaction. Each system illustrates a different role played by magnetic particle rotational motion. In the case of shear flow, particles are rotated out of alignment with magnetic field that is stationary, implying that field force is reduced and the state of magnetic stress is asymmetric. In oscillatory field, dissipative heating is generated by particles that rotate equally clockwise and counterclockwise while following the changing direction of the applied field; the state of stress is symmetric. Finally, in magnetic fluid subjected to rotating magnetic field, a state of asymmetric stress is induced that generates dissipative heating. The dissipative heating is due to coherent particle rotation plus magnetic relaxation. The study illustrates the compatibility of magnetic-stress-tensor and magnetic-field-energy descriptions of conditions in un-equilibrated magnetic fluid.

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