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Hydrodynamics of a horizontally rotating thin magnetizable liquid film

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Abstract

The axisymmetric thin magnetizable liquid film formed on a horizontally spinning disk in a magnetic field is considered. Analytical formulas for the steady and unsteady forms of the film surface are obtained. It is shown that the magnetic field affects the film surface form. It is found that by using a magnetic field a film of constant thickness can be obtained in spin coating processes.

1. Introduction

Creation of a thin liquid film of uniform thickness is of practical importance to many processes. A particular example is to be found in the production of floppy magnetic disks. A thin film of a ferrolacquer on the floppy disk is generated on a horizontal disk by spinning. Many authors considered the problem of the creation the thin film in absence of a magnetic field [1–3]. The hydrodynamic instability which occurs at the spreading edge of a thin wetting film was considered in Ref. [1]. In Refs. [2,3] instability of a rotating liquid film was discussed.

The theoretical study of the hydrodynamics of a horizontally rotating thin non-magnetizable liquid film in the case of sufficiently thin film has been considered by Needham and Merkin [1]. In Ref. [1] the asymptotic structure of the steady film was obtained; it was found that under certain conditions the steady film became unstable.

Here we consider the flow of a viscous magnetizable liquid (for example, a magnetic fluid) on a horizontal disk rotating in an arbitrary magnetic field. We do not consider a wetting process. Our purpose is to determine the effect of the magnetic field on the film surface.

The problem of the calculation of film velocity and film thickness in the steady and unsteady cases is solved for sufficiently thin films. It is shown that the effect of the magnetic field on the film surface and on time needed for generation of the steady film surface may be considerable. For example, the horizontal surface of a thin film may be created by using a magnetic field.

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2. Setting of the problem

Let us consider the axisymmetric flow of an incompressible viscous (ν is the kinematical viscosity) magnetizable liquid on a horizontal disk, rotating with an angular speed Ω. The magnetic field is axisymmetric and non-uniform the magnetic permeability is assumed to be uniform, $\mu =$ constant. The liquid is injected onto the disk at a specified flow rate through a small gap of height a at the bottom of a cylindrical reservoir of radius l situated at the center of the disk. Let us introduce the cylindrical polar coordinates r, θ, z, in the frame of reference rotating with the disk (r is the distance from the axis of the disk, z is the distance from the upper surface of the disk).

Within the rotating frame of reference the equations of the motion are the Navier–Stokes equations with inclusion of terms accounting for the centripetal and Coriolis forces. In the case of $\mu =$ constant, body magnetic forces are absent. Magnetic forces act on the free film surface with $z = \mathscr{D}(r, t)$.

The boundary conditions at the free film surface are the usual kinematic and dynamic conditions with terms which include the magnetic field strength and the magnetic permeability jump. The boundary condition for the velocity at the upper surface of the disk is $\boldsymbol{v} = 0$, $\boldsymbol{v} = (u, v, w)$. Let us assume that the liquid is injected onto the disk for $r = l$ with uniform velocity $u = \alpha(t)U_0$, $v = 0$, $w = 0$, $U_0 = \Omega^2 l a^2/\nu$ so the volume flux Q per unit length of perimeter is equal to $a\alpha(t)U_0$.

The initial conditions determine the velocity v and the film thickness $h=\mathscr{D}/a$ at $t=0$: $h(t=0)=H(r)$, $v(t=0)=V(r, z)$.

Let us introduce dimensionless variables and parameters:

$$r^*=\frac{r}{l},\quad z^*=\frac{z}{a},\quad p^*=\frac{p}{p\Omega^2 l^2},\quad h^*=\frac{D}{a},$$

$$u^*=\frac{u\nu}{l\Omega^2 a^2},\quad v^*=\frac{v\nu^2}{l\Omega^3 a^4},\quad w^*=\frac{w\nu}{\Omega^2 a^3},$$

$$B_n^{2*}=\frac{B_n^2}{p\Omega^2 l^2},\quad H_t^{2*}=\frac{H_t^2}{p\Omega^2 l^2},\quad \mathrm{Re}^*=\frac{\Omega^2 l a^3}{\nu^2},$$

$$\epsilon=\frac{a}{l},\quad F=\frac{\Omega^2 l}{g},\quad t^*=\frac{t}{t_0},\quad t_0=\frac{\nu}{a^2\Omega^2}.$$

Here p is the pressure of liquid, B_n, H_t are the projection of the magnetic induction on the normal to the film surface and the projection of the magnetic field strength on the plane tangent to the film surface, Re is the Reynolds number, F is the Froude number.

The dimensionless equations of motion are obtained in the form (henceforth asterisks are omitted)

$$\frac{1}{r}\frac{\partial}{\partial t}(ru)+\frac{\partial w}{\partial z}=0, \tag{1}$$

$$\epsilon\ \mathrm{Re}\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+\frac{\partial w}{\partial z}\right)$$
$$=-\frac{\partial p}{\partial r}+r+2\epsilon v\ \mathrm{Re}+\epsilon^2\ \mathrm{Re}^2\frac{v^2}{r}+\frac{\partial^2 u}{\partial z^2}$$
$$+\epsilon^2\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{u}{r^2}\right), \tag{2}$$

$$\epsilon\ \mathrm{Re}\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial r}+w\frac{\partial w}{\partial z}+\frac{uv}{r}\right)$$
$$=-2u+\frac{\partial^2 v}{\partial z^2}+\epsilon^2\left(\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}-\frac{v}{r^2}\right), \tag{3}$$

$$\epsilon^3\ \mathrm{Re}\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}\right)$$
$$=-\frac{\partial p}{\partial z}+\epsilon^2+\frac{\partial^2 w}{\partial z^2}-\frac{\epsilon}{F}+\epsilon^4\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right). \tag{4}$$

The dimensionless boundary conditions on the film surface $z=h(r, t)$ and on the upper plane of the disk $z=0$ are:
at $z=h(r, t)$, $r>1$:

$$\frac{\partial h}{\partial t}+u\frac{\partial h}{\partial r}-w=0, \tag{5}$$

$$2\epsilon^2\frac{\partial h}{\partial r}\left(\frac{\partial w}{\partial z}-\frac{\partial u}{\partial r}\right)+\left(\frac{\partial u}{\partial z}+\epsilon^2\frac{\partial w}{\partial r}\right)\left(1-\epsilon^2\left(\frac{\partial h}{\partial r}\right)^2\right)$$
$$=0, \tag{6}$$

$$\frac{\partial v}{\partial t}-\epsilon^2 r\frac{\partial h}{\partial r}\frac{\partial}{\partial r}\left(\frac{v}{r}\right)=0, \tag{7}$$

$$-p+2\epsilon^2\left[1+\epsilon^2\left(\frac{\partial h}{\partial r}\right)^2\right]^{-1}\left(\epsilon\frac{\partial u}{\partial r}\left(\frac{\partial h}{\partial r}\right)^2\right.$$
$$\left.+\frac{\partial w}{\partial z}-\frac{\partial u}{\partial r}\frac{\partial h}{\partial r}-\epsilon^2\frac{\partial w}{\partial z}\frac{\partial h}{\partial r}\right)$$
$$-\epsilon^2 T\frac{\partial^2 h}{\partial r^2}\left[1+\epsilon^2\left(\frac{\partial h}{\partial r}\right)^2\right]^{-1/2}+\frac{B_n^2}{8\pi}\left(\frac{1}{\mu}-1\right)$$
$$-\frac{H_t^2}{8\pi}(\mu-1)=0; \tag{8}$$

at $z=0$, $r>1$:

$$u=v=w=0; \tag{9}$$

at $0<z<1$, $r=1$:

$$u=\alpha,\quad v=w=0,\quad h=1. \tag{10}$$

Here $T=\gamma/(\Omega^2 la^2)$ is the Weber number, γ is the coefficient of surface tension of the liquid, $\alpha=\nu Q/(\Omega^2 la^3)=Q/U_0 a$.

We consider solutions of Eqs. (1)–(10) when $\epsilon\ll 1$ with $\mathrm{Re}=O(1)$ and F, α, $T=O(1)$ as $\epsilon\to 0$ [1]. It should be noted here that $\epsilon\ \mathrm{Re}=E^{-2}$, where $E=\nu/(\Omega a^2)$ is the Ekman number for the flow. We are considering flow with $E\gg 1$; this condition puts the restriction on the speed of rotation of the disk $\Omega\ll\nu/a^2$.

3. Solution

The solutions of Eqs. (1)–(4) at $\epsilon\ll 1$, $r=O(1)$, $t=O(1)$ may be determined as asymptotic expansions in the form ($A=(h, u, v, w, p)$)

$$A(r, z, t)=A_0(r, z, t)+\epsilon A_1(r, z, t)+\cdots \tag{11}$$

With $\epsilon\to 0$, in the case of $r=O(1)$, $t=O(1)$, we obtain to leading order the following solution of Eqs. (1)–(10):

$$p_0(r)=\frac{B_z^2}{8\pi}\left(\frac{1}{\mu}-1\right)-\frac{H_{xy}^2}{8\pi}(\mu-1),$$

$$u_0=(r-k(r))\frac{z}{2}(2h_0(r, t)-z),\quad k(r)=\frac{\partial p_0}{\partial r},$$

$$v_0=(r-k(r))\frac{z}{12}(4h_0(r, t)z^2-z^3-8h_0^3(r, t)), \tag{12}$$

$$w_0=\frac{z^3}{3}-z^2h_0(r, t)-\frac{z^2}{2}r\frac{\partial h_0}{\partial r}+k(r)\frac{\partial h_0}{\partial r}\frac{z^2}{2}$$
$$+\frac{z^2}{2}h_0(r, t)\left(\frac{\partial k(r)}{\partial r}+\frac{k(r)}{r}\right)$$
$$-\frac{z^3}{6}\left(\frac{\partial k(r)}{\partial r}+\frac{k(r)}{r}\right).$$

Here we use $B_n \simeq B_z$, $H_t \simeq H_{xy}$ at $\epsilon \to 0$, where B_z, H_{xy} are the projection of the magnetic induction on the z-axis and the projection of the magnetic field strength on the xy plane with $z = 0$. The distortion of the magnetic field by the magnetic fluid is neglected. The magnetic field ($|H_{xy}|$ and B_z), p_0 and k are consired to be known functions of r.

On substituting for u_0 and w_0 from Eq. (12) into condition (5) the equation determining $h_0(r, t)$ is obtained:

$$\frac{\partial h_0}{\partial t} + h_0^2 \frac{\partial h_0}{\partial r}(r - k) + \frac{2}{3} h_0^3 - \frac{h_0^3}{3}\left(\frac{\mathrm{d}k}{\mathrm{d}r} + \frac{k}{r}\right) = 0. \quad (13)$$

Eq. (13) differs from the equation for h_0 which was obtained in Ref. [1]. The function $k(r)$ and the magnetic field are related by Eq. (12). Here we assume that $k(r) < r$.

In the steady case the solution of Eq. (13) has the form (A is a constant):

$$h_0 = AM(r), \quad (14)$$

where

$$M(r) = \exp\left(\int_1^r \frac{-\frac{2}{3} + \frac{1}{3}(\partial k/\partial \xi + k/\xi)}{\xi - k(\xi)} \, \mathrm{d}\xi\right).$$

In the unsteady case Eq. (13) may be solved via the method of characteristics. Let us introduce the new variables $x = \ln(r)$, $\varphi = h^2$. The solution determined by the initial conditions along the characteristic

$$t = \int_\xi^x \varphi^{-1}(\eta, \xi)(\mathrm{e}^\eta - k(\eta))^{-1} \mathrm{e}^\eta \, \mathrm{d}\eta$$

is obtained as

$$\varphi(x, \xi) = G^2(\xi) \exp \int_\xi^x f(x) \, \mathrm{d}x,$$

$$f(x) = \frac{2(\mathrm{d}k/\mathrm{d}x + k) - 4\mathrm{e}^x}{3(\mathrm{e}^x - k)}.$$

The solution determined by the boundary conditions along the characteristic

$$t - \tau = \int_0^x \varphi^{-1}(\eta, \tau)(\mathrm{e}^\eta - k(\eta))^{-1} \mathrm{e}^\eta \, \mathrm{d}\eta$$

is obtained as:

$$\varphi(x, \tau) = A^2(\tau) \exp \int_0^x f(x) \, \mathrm{d}x.$$

It is obvious that the following conditions for h are valid: $h(x, 0) = G(x)$, $h(0, t) = A(t)$. Determining the dependence of the functions G and A on the true initial and boundary conditions is a problem. This problem may be solved by matching the solution in the outer region where $t \simeq O(1)$ and $r \simeq O(1)$ with the solution in the inner regions where $t \simeq O(\epsilon \mathrm{Re})$ or $r \simeq 1 + O(\epsilon)$. The method of matching leads to $A(t) = [3\alpha(t)/(1 - k(1)]^{1/3}$, $G(x) = H(\mathrm{e}^x)$.

4. Discussion

In the absence of a magnetic field, the film thickness is determined by the formula $h = Ar^{-2/3}$ for a steady flow. Therefore, we cannot create a film of constant thickness in a steady process. Consequently, in the non-magnetic case a film of constant thickness may be obtained only for unsteady flow when $Q = Q(t)$.

In a magnetic field the equality $f(x) = 0$ may be valid. This equality is valid if and only if $k = r + C/r$, $C =$ constant. It is obvious that the inequality $k < r$ is true for $C < 0$. Later on we consider the case of $C = -|C|$. In this case the solution (14) shows that the steady film surface is horizontal: $h = (3\alpha/|C|) =$ constant ($\alpha =$ constant).

Thus the steady film has a horizontal surface when the magnetic field has the configuration for which $k = r - |C|/r$. The film thickness decreases with an increase of $|C|$. Using the characteristic equation we can demonstrate that the time necessary for generating the steady film surface decreases with an increase of $|C|$. This means that a thin film with a fixed surface may be obtained using the magnetic field. This result is of practical importance to many processes arising in the field of chemical engineering.

References

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