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Model of a magnetizable elastic material

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Abstract

A model of a paramagnetic isotropic elastic material with small Young's modulus is proposed, wherein the stress tensor contains additional terms involving products of strain and field components. The new 'cross' terms produce an effective Young's modulus which depends on the magnetic field, whereas the classical model does not predict an influence of a magnetic field on deformation. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

A model describing field induced stresses in paramagnetic elastic isotropic materials may be proposed as the analogy of a previous model of elastic isotropic dielectrics in an electric field [1]. The main assumption of the model [1] is that a tensor of dielectric permeability depends linearly on a strain tensor u_{ij}. Using the same assumption about magnetic permeability we have calculated the free energy of paramagnetic elastic isotropic materials in an external magnetic field. Our expression for the free energy has additional 'cross' terms (terms containing products of the magnetic field and the strain tensor components). Analogical terms (terms containing products of the electric field and the strain tensor components) are neglected in the model [1]. Same 'cross' terms occur in an expression for the free energy of elastic anisotropic ferromagnetic materials [2]. However, for rigid ferromagnetic materials (small deformation and large Young's modulus) additional 'cross' terms are small. As it is proven hereinafter the new model (with additional 'cross' terms) describes equilibriumly magnetizable (paramagnetic) media with small Young's modulus when deformations are large. For example, the model may be valid for isotropic magnetizable composite materials on the basis of nonrigid elastomers or rubbers. It should be noted that this model may be valid for other paramagnetic homogeneous materials with small Young's modulus.

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Here we consider the effect of the additional 'cross' terms on the stress tensor. The contribution of the 'cross' terms to the stress tensor of an elastic material in a magnetic field can be described by effective values of the Young's modulus that depend on the magnetic field. Here we do not describe the well-known ΔE-effect for anisotropic ferromagnetic materials. The model predictions for two examples of deformation of material in an applied magnetic field are compared with the previous model (without 'cross' terms) [1] in which there is no influence of the applied field on the elastic response.

2. Stress tensor of magnetizable elastic materials in a magnetic field

Let us consider a equilibriumly magnetizable (paramagnetic) elastic isotropic material with a magnetic permeability tensor μ_{ij} that depends on the strain tensor components u_{ij},

$$\mu_{ij}(u_{kl}) = \mu_0(\mu^0 g_{ij} + a_1 u_{ij} + a_2 u_k^k g_{ij}), \tag{1}$$

where $\mu_0(=4\pi\times10^{-7}\,\mathrm{NA}^{-2})$ is the permeability of free space, g_{ij} is the metric tensor, μ^0 is the permeability of the nondeformed material, and a_1 and a_2 are scalar functions of thermodynamic parameters. The parameters a_1 and a_2 can be estimated for isotropic magnetizable composite media through the analogy with heterogeneous dielectric materials [3]

$$a_1 = -\tfrac{2}{5}(\mu^0 - 1)^2,$$

$$a_2 = -(\mu^0 - 1)\left(\tfrac{7}{15}\mu^0 + \tfrac{8}{15}\right). \tag{2}$$

The thermodynamic potential of the material in a magnetic field $\boldsymbol{H}$ is $\tilde{F} = U - TS - \boldsymbol{H}\cdot\boldsymbol{B}$, where U and S are the internal energy and the entropy of the medium and field per unit volume, T is the temperature, and $\boldsymbol{B}$ is the magnetic induction. According to Gibbs' identity, we can represent the free energy as

$$\tilde{F} = F_0(H=0) - \int_0^{\boldsymbol{H}} \boldsymbol{B}\cdot \mathrm{d}\boldsymbol{H},\ B_i = \mu_{ij}H^j. \tag{3}$$

The free energy $F_0(H=0)$ is represented by Hooke's law,

$$F_0(H=0) = 0.5\lambda(u_k^k)^2 + \eta u_{ij}u^{ij}, \tag{4}$$

where λ and η are the Lamé coefficients. Substituting Eq. (1) into Eq. (3), the thermodynamic potential $\tilde{F}$ becomes

$$\tilde{F} = F_0 - \mu_0\mu^0\frac{H^2}{2} - \mu_0 a_2 u_i^i\frac{H^2}{2} - \mu_0 a_1 u_{ij}\frac{H^iH^j}{2}. \tag{5}$$

The stress tensor in a magnetizable elastic medium in the presence of a magnetic field may be obtained by analogy with the development in Ref. [1],

$$p_{ij} = \tilde{F}g_{ij} + \left.\frac{\partial\tilde{F}}{\partial u_{ij}}\right|_{T,\boldsymbol{H}} + H_iB_j. \tag{6}$$

Upon substituting the expression for the free energy, Eq. (5), into Eq. (6), the stress tensor p_{ij} becomes

$$\begin{aligned} p_{ij} &= \lambda u_k^k g_{ij} + 2\eta u_{ij} \\ &\quad + \mu_0\frac{2\mu^0 - a_1}{2}H_iH_j - \mu_0\frac{\mu^0 + a_2}{2}H^2 g_{ij} \\ &\quad + \mu_0 a_1\left(u_{jk}H_iH^k - u_{kl}\frac{H^kH^l}{2}g_{ij}\right) \\ &\quad + \mu_0 a_2 u_l^l\left(H_iH_j - \frac{H^2}{2}g_{ij}\right). \end{aligned} \tag{7}$$

The products of the strain tensor and magnetic field components ('cross' terms in Eq. (7)) were not accounted for in the model developed in Ref. [1].

3. Deformation of materials in a magnetic field

We will consider deformation of an elastic body which is a parallelepiped in a Cartesian coordinate system with axes directed along the edges of the parallelepiped when a uniform magnetic field $\boldsymbol{H}_0$ is applied ($\boldsymbol{H}_0$ is a magnetic field without the body). A uniform surface force with density $\boldsymbol{F} = \pm F\boldsymbol{e}_z$ is applied to the xy faces of the parallelepiped ($\boldsymbol{e}_i$ is the base vector along the i-axis). The other surfaces of the parallelepiped are stress-free.

We assume that the magnetic field $\boldsymbol{H}=\boldsymbol{H}_0$ is a uniform one everywhere ($H_0 \gg M$, where $\boldsymbol{M}$ is a vector of magnetization of the material, noninductive approximation). The stress tensor components satisfy the equations of statics, because we search for uniform deformations and $\boldsymbol{H}$ is a uniform one. The boundary conditions on the parallelepiped surfaces Σ_{yz}, Σ_{zx}, and Σ_{xy} are

$$\{p_{11}\}|_{\Sigma_{yz}}=0, \quad \{p_{22}\}|_{\Sigma_{zx}}=0, \quad \{p_{33}\}|_{\Sigma_{xy}}=F. \tag{8}$$

Here $\{A\}=A_m-A_a$, with indices m and a denoting values of A on the material and air sides of the surfaces, respectively. Our objective is to determine the deformation of the material in the uniform magnetic field caused by the external surface force density F. Below we consider the two different cases, $\boldsymbol{H}_0=H_0\boldsymbol{e}_x$ and $\boldsymbol{H}_0=H_0\boldsymbol{e}_z$.

H*₀ *parallel to the x-axis: Taking into account expression (7), we write down Eq. (8) in the form

$$\lambda u_l^l+2\eta u_{11}+\mu_0(a_2u_l^l+a_1u_{11})\frac{H_0^2}{2}=A,$$

$$\lambda u_l^l+2\eta u_{22}-\mu_0(a_2u_l^l+a_1u_{11})\frac{H_0^2}{2}=B,$$

$$\lambda u_l^l+2\eta u_{33}-\mu_0(a_2u_l^l+a_1u_{11})\frac{H_0^2}{2}=F+B,$$

$$A(H_0)=\mu_0(1-\mu^0+a_2+a_1)\frac{H_0^2}{2},$$

$$B(H_0)=\mu_0(\mu^0-1+a_2)\frac{H_0^2}{2}. \tag{9}$$

Taking into account Eq. (9), the expression for the strain u_{33} is obtained:

$$u_{33}=\frac{\lambda+\eta}{\eta(3\lambda+2\eta)}\left(\frac{(F+2B)(1+k_1H_0^2)}{(1-k_2H_0^2+k_3H_0^2)}+\frac{A(1+2k_1H_0^2)}{(1-k_2H_0^2+k_3H_0^2)}\right)-\frac{B+A}{2\eta},$$

$$k_1=\frac{\mu_0a_1}{4\eta},\ k_2=\frac{\mu_0a_2}{2(3\lambda+2\eta)},$$

$$k_3=\frac{\mu_0a_1(2\lambda+\eta)}{2\eta(3\lambda+2\eta)}. \tag{10}$$

In the classical model [1] k_1, k_2, k_3 equal zero. Assuming $F \gg A$ and $F \gg B$, we obtain the following expression for u_{33}:

$$u_{33}=\frac{\lambda+\eta}{\eta(3\lambda+2\eta)}\frac{F(1+k_1H_0^2)}{(1-k_2H_0^2+k_3H_0^2)}. \tag{11}$$

Thus if the extending force is sufficiently large, then the influence of the magnetic field can be described by the introduction of the effective value $E_{\rm y}^{\rm eff}$ of the Young's modulus in the magnetic field as follows:

$$E_{\rm y}^{\rm eff}=E_y\frac{1-k_2H_0^2+k_3H_0^2}{1+k_1H_0^2}, \tag{12}$$

$$E_{\rm y}=\frac{\eta(3\lambda+2\eta)}{\lambda+\eta}. \tag{13}$$

Assuming $\lambda \gg \eta$, the expression for u_{33} follows from Eq. (11):

$$u_{33}=F/E_{\rm y}^{\rm eff},\ E_{\rm y}^{\rm eff}=E_{\rm y}\frac{1+2/3G}{1+1/2G},$$

$$E_y=3\eta,\ G=\mu_0a_1H_0^2/2\eta=2k_1H_0^2.$$

Thus if the extending force is sufficiently large and $\lambda \gg \eta$, the effective value of the Young's modulus in the magnetic field $E_{\rm y}^{\rm eff}$ depends essentially on the field if $|G|$ is not small. The classical model does not predict any influence of the magnetic field on deformation for this case.

H*₀ *parallel to the z-axis: In this case, the boundary conditions (8) yield

$$\lambda u_l^l+2\eta u_{11}-\mu_0(a_2u_l^l+a_1u_{33})\frac{H_0^2}{2}=B,$$

$$\lambda u_l^l+2\eta u_{22}-\mu_0(a_2u_l^l+a_1u_{33})\frac{H_0^2}{2}=B,$$

$$\lambda u_l^l+2\eta u_{33}+\mu_0(a_2u_l^l+a_1u_{33})\frac{H_0^2}{2}=F+A,$$

From these equations we can calculate u_{33} as follows:

$$u_{33}=\frac{-(2\lambda+\eta)}{\eta(3\lambda+2\eta)}\left(\frac{(F+A)(1+2k_1H_0^2)}{1-k_2H_0^2+k_3H_0^2}+\frac{2B(1+k_1H_0^2)}{1-k_2H_0^2+k_3H_0^2}\right)+\frac{F+B+A}{\eta}.$$

Consider the case of large forces: $F \gg A$ and $F \gg B$. Then the deformation can be determined by the formula

$$u_{33} = \frac{-(2\lambda + \eta)}{\eta(3\lambda + 2\eta)} \frac{F(1 + 2k_1 H_0^2)}{1 - k_2 H_0^2 + k_3 H_0^2} + \frac{F}{\eta}.$$

When the extending force is sufficiently large and $\lambda \gg \eta$, the influence of the magnetic field can be described by the introduction of the effective value E_y^{eff} of the Young's modulus in the magnetic field is as follows:

$$E_y^{\text{eff}} = E_y \left(1 + \tfrac{2}{3} G\right).$$

The examples of deformation of the material considered above illustrate that for large forces the classical model predicts no influence of the magnetic field on deformation while the new model in this case allows us to introduce the effective value of the Young's modulus, which depends on the field. Obviously, the 'cross' terms represented in the formula for the stress tensor (7) should be taken into account when the deformation of materials is not small and the Lamé coefficient η is small enough: $|G| = |\mu_0 a_1 H_0^2 / 2\eta| \sim 1$. So this model can describe the properties of isotropic composite materials that are created on the basis of nonrigid polymers or rubbers.

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