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Stability Properties of Ferromagnetic Fluids in the Presence of an Oblique Field and Mass and Heat Transfer

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Abstract

An analysis of the stability of a basic flow of streaming magnetic fluids in the presence of an oblique periodic magnetic field is made. The particular investigated profile is the classical Kelvin-Helmholtz profile modified by the addition of the influence of mass and heat transfer across the interface. In the case of a uniform field, the dispersion relation is obtained and discussed both analytically and numerically and the stability diagrams are also obtained. It is found that the effect of the field depends strongly on the choice of some physical parameters of the system. For a time dependent field, the characteristic values and intervals of stability are investigated through Hill's differential equation. The special case of an oscillatory field is governed by the Mathieu equation, which is also discussed. The presence of mass and heat transfer resulted in a damped Mathieu equation. The analysis near the marginal state as well as Rayleigh-Taylor instabilities are discussed. Regions of stability and instability are identified. It is found that the mass and heat transfer parameter has a destabilizing influence regardless of the mechanism of the field. It is also found that the field frequency plays a dual role in the stability picture.

1. Introduction

Magnetic fluids, also known as ferrofluids, are ultrastable colloidal suspensions of subdomain ferro- or ferri-magnetic particles-e.g., magnetite (Fe_3O_4)-dispersed in various carrier liquids (Rosensweig [1]). These materials behave like a quasi-homogeneous strongly magnetizable liquid due to the presence of approximately 10^{17}–10^{18} magnetic particles in one cubic centimetre. Many experimental results confirmed that colloidal particles in magnetic fluids coagulate and form chain clusters as a result of their mutual interactions, this process being enhanced in the presence of a magnetic field. The chain formation process, together with the reorientation of individual particles in the presence of a magnetic field, are responsible for the anisotropy of the physical properties of the magnetic fluids [2]. For example, magneto-optical effects induced in thin magnetic-fluid layers are well explained by the above-mentioned microstructural process [3]. The sound velocity and the acoustic-attenuation coefficient in magnetic fluids are also dependent on the angle between the sound propagation direction and the external magnetic field.

The instability of the plane interface separating two Newtonian fluids when one is accelerated towards the other or when one is superposed over the other has been studied by several authors. Chandrasekhar [4] has given a detailed account of these investigations. The model for the classical Kelvin-Helmholtz instability involves a horizontal interface between two fluids with different parallel, uniform, horizontal velocities. This instability, which arises as a consequence of a relative drift velocity of two fluids along the surface of discontinuity, has great relevance to various physical phenomena such as cometary tails and the magnetospheric boundary. The Kelvin-Helmholtz instability due to shear flow in stratified fluids has attracted the attention of many researchers because of its determinant influence on the stability of planetary and stellar atmosphers and in practical applications. The study of the Kelvin-Helmholtz instability has a long history in hydrodynamics. It is well known that in two-dimensional inviscid, incompressible hydrodynamics, there are two invariants of fluid motion, i.e., the total kinetic energy and the enstrophy (mean square vorticity). The existence of these two invariants requires that, in two-dimensional inviscid incompressible hydrodynamics, the energy cascades to long wavelength or vortices with similarly signed vorticity must tend to group together [5]. Indeed, hydrodynamical experiments have shown at the late stage of the Kelvin-Helmholtz instability two vortical structures combine to form a single, larger vortical structure. Such a nonlinear evolution of the Kelvin-Helmholtz instability has been reproduced by numerical experiments and theoretical investigations [6].

The linear formulation of the Kelvin-Helmholtz instability in the context of magnetic fluids was investigated by Rosensweig [1]. His analysis revealed that the velocity difference that can be supported by the fluids before the instability sets in is enhanced if the difference in the permeabilities of the fluids across the interface and the strength of the applied magnetic field are increased. These fluids differ from magnetohydrodynamic fluids since no electric current flows in these fluids. Because of the wide range of important industrial applications, there has been a growing interest in the study of magnetic fluids when subjected to normal and tangential magnetic fields. The propagation of plane waves in magnetic fluids in the presence of a tangential magnetic field has been investigated theoretically as well as experimentally by Zelazo and Melcher [7]. These authors have demonstrated that the magnetic field exerts a stabilizing influence on the stability of the fluid surface. In their experiment, a plane wave of specific wavelength, consistent with the boundary conditions, was imposed on the interface, and the subsequent frequency shift for various strengths of the magnetic field was measured. Both theoretical and experimental results show an upward shift of frequency of the imposed wavelength as a function of the magnetic field. Cowley and Rosensweig [8] reported that an instability sets in when the applied magnetic field, which is normal to the fluid surface, exceeds the critical magnetic field. Their pioneer experiment demonstrated the existence of an instability leading to the appearance of regular hexagonal cells. In their investigation of the nonlinear evolution of wave packets on the surface of a magnetic fluid, Malik and Singh [9] showed that the wave train solution of constant amplitude is unstable against modulation if the product of the group velocity rate and the nonlinear interaction coefficient is negative. Elhefnawy [10] studied the nonlinear evolution of ferromagnetic fluids in the presence of an oblique magnetic field. He found that the stability of the system depends on the direction of the magnetic field.

The phenomenon of parametric resonance arises in many branches of physics and engineering. One of the important problems is that of dynamic instability which is the response of mechanical and elastic systems to time-varying loads, especially periodic loads. There are cases in which the introduction of small vibrational loading can stabilize a system which is statically unstable or destabilize a system that is statically stable. Faraday [11] first studied experimentally the patterns of standing waves in a vessel subjected to vertical oscillation. He found that the frequency of the surface oscillation was one half of that of the external forcing. Benjamin and Ursell [12] explained the excitation of standing waves of an inviscid liquid, which is associated with the instability of the Mathieu equation for a parametric resonant mode. Kelly [13] considered the effect of an oscillatory component in the basic velocity on the stability of the classical Kelvin-Helmholtz profile. There are some physical situations when one needs a limited band of wave numbers to achieve instability. For example, in biophysics, Zimmerman [14] showed that the cell membrane is formed by a number of adjacent cells if they are subjected to a periodic field. Also, the membrane breaks down if a field, at a given strength, is applied to it. The treatment of parametric excitation systems having many degrees of freedom and distinct natural frequencies is usually operated by using the multiple time scales as given by Nayfeh [15]. The behavior of such system is described by an equation of the Hill or Mathieu types [16,17]. It is well known that the stability of such solutions may be described by means of the characteristic curves of Mathieu functions which admit regions of resonance instability. Bashtovoi and Rosensweig [18] have reported the parametric excitation of surface waves in a cylindrical vessel containing a magnetic fluid when the externally applied magnetic fluid acts normally to the fluid surface. In their experiment, the surface waves get excited when the magnetization field is less than the critical one. An interesting experimental observation of period doubling in the normal field instability problem has been observed by Bacri *et al.* [19].

On the other hand, the mechanism of mass and heat transfer across an interface is of great importance in numerous industrial and environmental processes. These include the design of many types of contacting equipment, e.g. boilers, condensers, evaporators, gas absorbers, piplines, chemical reactors, nuclear reactors, and in other problems such as the aeration of rivers and the sea. In most cases of practical importance, the liquid is turbulent and the transport across the gas-liquid interface is governed by the liquid side. In early investigations, Hsieh [20] formulated the general problem of interfacial fluid flow with mass and heat transfer and applied it to discuss the Kelvin-Helmholtz instability problem. In nuclear reactor cooling [21] of fuel rods by liquid coolants, the geometry of the system is in many cases cylindrical. Therefore, Nayak and Chakraborty [22] studied the problem of Kelvin Helmholtz stability with mass and heat transfer in cylindrical geometry using Hsieh's simplified formulation and compared their results with those in plane geometry. The effect of a magnetic field on the stability of cylindrical flow with mass and heat transfer has been investigated by Elhefnawy and Radwan [23]. They found that the instability criteria is independent of mass and heat transfer coefficients, but it is different from that in the same problem without heat and mass transfer.

The classical Rayleigh-Taylor problem deals with the stability of a heavy fluid supported by a ligher one. Because of gravity, the former is accelerated in the direction of the later. In the absence of main flow or shear, the Rayleigh-Taylor instability is left as the source of perturbation growth in the case of an unsteady stratified medium. Recently, new advances in Rayleigh-Taylor instability for non-dissipative incompressible fluids have been made in which a continuously stratified flow is subjected to a general stratified horizontal magnetic field [24].

The aim of this paper is to study the stability and instability conditions of a Kelvin-Helmholtz problem of a liquid layer over a vapour layer of finite depth, in the presence of an oblique time-dependent magnetic field. The transfer of mass and heat across the interface is taken into account. The effect of nonlinearity on the problem at hand will not be discussed here but will be the subject of a subsequent paper.

2. Formulation of the problem

We shall study two-dimensional progressive waves at the interface $z = 0$, which separates two incompressible inviscid magnetic fluids. Without any loss of generality, the cartisian coordinates (x, z) are taken into consideration, where the surface wave propagates in the x-direction and gravity g acts in the negative z-direction. The x-axis is the mean level of the wave. The fluids are bounded by horizontal planes at $z = h_2$ and $z = -h_1$. The subscripts 1 and 2 refer to the lower and upper fluids, respectively. The fluid of density ρ_1 with magnetic permeability μ_1 occupies the region $-h_1 < z < 0$, and the fluid of density ρ_2 with magnetic permeability μ_2 is in the region $0 < z < h_2$. The basic unperturbed flow has a constant velocity u_1 in the x-direction in the lower layer $(-h_1 < z < 0)$ and a constant velocity u_2 in the upper layer $(0 < z < h_2)$. The temperatures at $z = -h_1$, $z = h_2$ and $z = 0$ are taken as T_1, T_2 and T_0, in order, where $T_1 > T_0 > T_2$.

The two fluids are influenced by an oblique time dependent-magnetic field,

$$\boldsymbol{H}_j = H_j(t)(\cos\theta_j \boldsymbol{e}_x + \sin\theta_j \boldsymbol{e}_z), \quad j = 1, 2 \tag{2.1}$$

where $\boldsymbol{e}_x$ and $\boldsymbol{e}_z$ are unit vectors in the x and z-directions, respectively, and θ_j is the angle between the field $\boldsymbol{H}_j$ and the x-axis.

We shall assume that there are no free currents at the surface of separation in the equilibrium state. Therefore, the tangential component of the field is continuous at the interface, while the normal one is discontinuous by the ratio of the permeabilities, i.e.

$$H_1(t)\cos\theta_1 = H_2(t)\cos\theta_2, \tag{2.2}$$

and

$$\mu_1 H_1(t)\sin\theta_1 = \mu_2 H_2(t)\sin\theta_2. \tag{2.3}$$

We also consider all "functions $H_1(t)H_2(t)$ of class P" which are defined by

$$\left[\pi\int_0^\pi |H_1(t)H_2(t)|^P \mathrm{d}t\right]^{\frac{1}{P}} = 1, \tag{2.4}$$

where $P = 1, 2, 3, \ldots$, or $P = \infty$. If $P = \infty$ Eq. (2.4) means that

$$\max|H_1(t)H_2(t)| = 1. \tag{2.5}$$

We assume that $H_1(t)H_2(t)$ is continuous except for a finite number of points, where $H_1(t)H_2(t)$ may have a jump.

The motion considered here is irrotational in both of the magnetic fluids and there exists a velocity potential ϕ. For an incompressible fluid, the potential ϕ satisfies the Laplace equation.

In a magneto-quasistatic system, with a negligible displacement current, Maxwell's equations in the absence of free currents are

$$\nabla_\Lambda \tilde{\boldsymbol{H}} = 0 \quad \text{and} \quad \nabla\cdot(\mu\tilde{\boldsymbol{H}}) = 0, \tag{2.6}$$

because there are no free currents. Therefore, the magnetic field is a curl free vector having magnetic scalar potential Ψ such that

$$\tilde{\boldsymbol{H}}_j = \boldsymbol{H}_j - \nabla\Psi_j, \quad j = 1, 2, \tag{2.7}$$

where $\boldsymbol{H}_j$ is given by Eq. (2.1).

It follows from Eqs (2.6) and (2.7) that the magnetic potential also satisfies the Laplace equation.

Thus, the equations governing the velocity potential ϕ and the magnetic potential Ψ are

$$\nabla^2\phi_1 = \nabla^2\Psi_1 = 0, \quad -h_1 < z < \eta(x, t), \tag{2.8}$$

$$\nabla^2\phi_2 = \nabla^2\Psi_2 = 0, \quad \eta(x, t) < z < h_2, \tag{2.9}$$

with the conditions

$$\left.\begin{aligned} &\frac{\partial\phi_1}{\partial z} = 0 && \text{at } z = -h_1, \\ &\frac{\partial\phi_2}{\partial z} = 0 && \text{at } z = +h_2, \\ &\frac{\partial\Psi_1}{\partial x} = 0 && \text{at } z = -h_1, \\ &\frac{\partial\Psi_2}{\partial x} = 0 && \text{at } z = +h_2, \end{aligned}\right], \tag{2.10}$$

where $z = \eta(x, t)$ denotes the elevation of the interface at time t.

The solutions for ϕ_j and Ψ_j $(j = 1, 2)$ have to satisfy the boundary conditions, so that, if we assume that the interface between the two magnetic fluids is given by $S(x, z, t) = z - \eta(x, t) = 0$, the linearized boundary conditions at the interface $z = \eta(x, t)$ are [1,20]:

(1) The conservation of mass across the interface is given as

$$\rho_1\left(\frac{\partial S}{\partial t} + \boldsymbol{v}_1\cdot\nabla S\right) = \rho_2\left(\frac{\partial S}{\partial t} + \boldsymbol{v}_2\cdot\nabla S\right)$$

$$\text{or} \quad \rho_1\left(\frac{\partial\phi_1}{\partial z} - \frac{\partial\eta}{\partial t} - u_1\frac{\partial\eta}{\partial x}\right) = \rho_2\left(\frac{\partial\phi_2}{\partial z} - \frac{\partial\eta}{\partial t} - u_2\frac{\partial\eta}{\partial x}\right), \tag{2.11}$$

where $\boldsymbol{v}_j = u_j\boldsymbol{e}_x + \nabla\phi_j$.

(2) The tangential component of the magnetic field must be continuous across the interface, i.e. $\boldsymbol{n}_\Lambda(\tilde{\boldsymbol{H}}_1 - \tilde{\boldsymbol{H}}_2) = 0$, or

$$\frac{\partial\Psi_1}{\partial x} - H_1(t)\sin\theta_1\frac{\partial\eta}{\partial x} = \frac{\partial\Psi_2}{\partial x} - H_2(t)\sin\theta_2\frac{\partial\eta}{\partial x}, \tag{2.12}$$

where $\boldsymbol{n}$ is the unit normal vector to the interface and is given by

$$\boldsymbol{n} = \frac{\nabla S}{|\nabla S|} \cong -\frac{\partial\eta}{\partial x}\boldsymbol{e}_x + \boldsymbol{e}_z. \tag{2.13}$$

(3) Since there are no free currents at the interface, the normal component of the magnetic induction vector is continuous at the interface, i.e. $\boldsymbol{n}.(\mu_1\tilde{\boldsymbol{H}}_1 - \mu_2\tilde{\boldsymbol{H}}_2) = 0$, or

$$\mu_1\left[\frac{\partial\Psi_1}{\partial z} + H_1(t)\cos\theta_1\frac{\partial\eta}{\partial x}\right] = \mu_2\left[\frac{\partial\Psi_2}{\partial z} + H_2(t)\cos\theta_2\frac{\partial\eta}{\partial x}\right]. \tag{2.14}$$

(4) The interfacial condition for the conservation of energy, yields

$$L_{\rho_1}\left(\frac{\partial S}{\partial t} + \boldsymbol{v}_1\cdot\nabla S\right) = F(z), \tag{2.15}$$

where L is the latent heat of transformation from the fluid of density ρ_1 to the fluid of density ρ_2 and $F(z)$ is a function of the instantaneous profile of the interface and is determined from the heat transfer relation at equilibrium [20].

In the equilibrium state, the heat fluxes in the direction of z increasing in the two regions 1 and 2 are $\frac{K_1(T_1 - T_0)}{h_1}$ and $\frac{K_2(T_0 - T_2)}{h_2}$ respectively, where K_1 and K_2 are the thermal conductivities of the two fluids. As in Hsieh [20], we denote

$$F(z) = \frac{K_2(T_0 - T_2)}{h_2 - z} - \frac{K_1(T_1 - T_0)}{h_1 + z}. \tag{2.16}$$

If we expand $F(z)$ about $z = 0$ by Maclaurin series expansion, one gets

$$F(z) = F(0) + \eta F'(0) + \cdots. \tag{2.17}$$

It is clear that $F(0)$ represents the net heat flux from the interface into the fluid regions. Since it is an equilibrium state, we have

$$F(0) = 0, \tag{2.18}$$

$$\text{so that} \quad \frac{K_2(T_0 - T_2)}{h_2} = \frac{K_1(T_1 - T_0)}{h_1} = G(\text{say}), \tag{2.19}$$

indicating that, in the equilibrium state, the heat fluxes are equal across the interface in the two fluids.

Substituting (2.16)–(2.19) into (2.15), we obtain

$$\rho_1\left(\frac{\partial\phi_1}{\partial z} - \frac{\partial\eta}{\partial t} - u_1\frac{\partial\eta}{\partial x}\right) = \alpha\eta, \tag{2.20}$$

where $\alpha = \frac{G}{L}\left(\frac{1}{h_1} + \frac{1}{h_2}\right)$.

Now, the vapor phase is usually hotter than the liquid one so α is always positive. If fluid (2) is a liquid and fluid (1) a vapor, then L and G are both positive since $T_1 > T_0 > T_2$. If fluid (1) is a liquid and fluid (2) a vapor, then L and G are both negative. Thus in both cases α is always positive. It is noteworthy that the effect of mass and heat transfer are revealed through a single positive parameter, α, in this simplified version. It would be interesting to see that the correlation of experimental data would be facilitated by this simplification.

(5) Finally, the dynamical condition that the normal stresses should be continuous across the interface, gives

$$\left[P_1 - P_2 + \sigma\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\right] n_i = [\tau_{ij1} - \tau_{ij2}] n_j, \tag{2.21}$$

where P is the pressure, σ is the surface tension coefficient, R_1 and R_2 are the two principal radii of curvature of the interface,

$$\tau_{ij} = \mu \tilde{H}_i \tilde{H}_j - \frac{1}{2}\mu \tilde{H}^2 \delta_{ij}, \tag{2.22}$$

is the Maxwell stress tensor, and n_i and n_j are components of the normal vector to the interface.

By eliminating the pressure by Bernoulli's equation, condition (2.21) can be rewritten as

$$\begin{aligned} g\eta(\rho_1 - \rho_2) + \rho_1 \frac{\partial \phi_1}{\partial t} - \rho_2 \frac{\partial \phi_2}{\partial t} + \rho_1 u_1 \frac{\partial \phi_1}{\partial x} - \rho_2 u_2 \frac{\partial \phi_2}{\partial x} \\ = \sigma \frac{\partial^2 \eta}{\partial x^2} + \mu_2 H_2(t)\left(\cos\theta_2 \frac{\partial \Psi_2}{\partial x} - \sin\theta_2 \frac{\partial \Psi_2}{\partial z}\right) \\ - \mu_1 H_1(t)\left(\cos\theta_1 \frac{\partial \Psi_1}{\partial x} - \sin\theta_1 \frac{\partial \Psi_1}{\partial z}\right) \end{aligned} \tag{2.23}$$

The solutions of Eqs (2.8) and (2.9) with the conditions (2.10), (2.11), (2.12), (2.14) and (2.20) for travelling waves with respect to the variable x that decays far from the interface are

$$\eta = \gamma(t) e^{ikx} + c.c., \tag{2.24}$$

$$\phi_1 = \frac{\cosh k(z + h_1)}{k \sinh kh_1}\left[\frac{d\gamma}{dt} + \left(iku_1 + \frac{\alpha}{\rho_1}\right)\gamma\right] e^{ikx} + c.c., \tag{2.25}$$

$$\phi_2 = \frac{\cosh k(z - h_2)}{k \sinh kh_2}\left[\frac{d\gamma}{dt} + \left(iku_2 + \frac{\alpha}{\rho_2}\right)\gamma\right] e^{ikx} + c.c., \tag{2.26}$$

$$\begin{aligned} \Psi_1 = &\left[\frac{\sinh k(z + h_1)}{(\mu_2 \sinh kh_1 \cosh kh_2 + \mu_1 \cosh kh_1 \sinh kh_2)}\right] \\ &\times \big[i \sinh kh_2(\mu_2 H_2(t)\cos\theta_2 - \mu_1 H_1(t)\cos\theta_1) \\ &-\mu_2 \cosh kh_2 (H_2(t)\sin\theta_2 - H_1(t)\sin\theta_1)\big]\gamma e^{ikx} + c.c., \end{aligned} \tag{2.27}$$

$$\begin{aligned} \Psi_2 = &\left[\frac{\sinh k(z - h_2)}{(\mu_2 \sinh kh_1 \cosh kh_2 + \mu_1 \cosh kh_1 \sinh kh_2)}\right] \\ &\times \big[i \sinh kh_1(\mu_2 H_2(t)\cos\theta_2 - \mu_1 H_1(t)\cos\theta_1) \\ &+\mu_1 \cosh kh_1 (H_2(t)\sin\theta_2 - H_1(t)\sin\theta_1)\big]\gamma e^{ikx} + c.c., \end{aligned} \tag{2.28}$$

where k is the wave number, $c.c.$ denotes the complex conjugate, and $\gamma(t)$ is an arbitrary function of time t and satisfies the following differential equation:

$$a_0 \frac{d^2\gamma(t)}{dt^2} + (a_1 + ib_1)\frac{d\gamma(t)}{dt} + (a_2 + ib_2)\gamma(t) = 0. \tag{2.29}$$

The coefficients in Eq. (2.29) are

$$a_0 = \rho_1 \coth kh_1 + \rho_2 \coth kh_2, \tag{2.30}$$

$$a_1 = \alpha(\coth kh_1 + \coth kh_2), \tag{2.31}$$

$$b_1 = 2k(\rho_1 u_1 \coth kh_1 + \rho_2 u_2 \coth kh_2), \tag{2.32}$$

$$\begin{aligned} a_2 = &k\left[g(\rho_1 - \rho_2) - k(\rho_1 u_1^2 \coth kh_1 + \rho_2 u_2^2 \coth kh_2) + \sigma k^2\right] \\ &+ \left[\frac{k^2}{\mu_1 \coth kh_1 + \mu_2 \coth kh_2}\right][[\mu_2 H_2(t)\cos\theta_2 \\ &- \mu_1 H_1(t)\cos\theta_1]^2 - \mu_1\mu_2[H_2(t)\sin\theta_2 - H_1(t)\sin\theta_1]^2 \\ &\times \coth kh_1 \coth kh_2], \end{aligned} \tag{2.33}$$

$$b_2 = \alpha k(u_1 \coth kh_1 + u_2 \coth kh_2). \tag{2.34}$$

3. Stability behaviour due to a uniform oblique magnetic field

The solution of Eq. (2.29) will decide the criterion of stability of the system. Accordingly, the system will be stable if the solution for $\gamma(t)$ remains bounded as $t \to \infty$, otherwise it is unstable.

A special case occurs when the applied magnetic fields are uniform, i.e. $H_1(t) \to H_1$ and $H_2(t) \to H_2$. In this case, we find that the coefficient a_2, Eq. (2.33), is independent of t. Therefore the solution of Eq. (2.29) becomes

$$\gamma(t) = \text{const.}\exp(-i\omega t), \tag{3.1}$$

where ω is the (complex) frequency of the disturbance. It is clear that the system is stable if the imaginary part of ω is either less than or equals zero.

Substituting (3.1) into (2.29) and using (2.3), we have

$$a_0\omega^2 + (-b_1 + ia_1)\omega + (a_{20} - ib_2) = 0, \tag{3.2}$$

where

$$\begin{aligned} a_{20} = &k[g(\rho_2 - \rho_1) + k(\rho_1 u_1^2 \coth kh_1 + \rho_2 u_2^2 \coth kh_2) - \sigma k^2] \\ &- k^2 H_1 H_2 (\mu_2 - \mu_1)^2 (\cos\theta_1 \cos\theta_2 \\ &- \sin\theta_1 \sin\theta_2 \coth kh_1 \coth kh_2) \\ &\times (\mu_1 \coth kh_1 + \mu_2 \coth kh_2)^{-1}. \end{aligned}$$

The study of the properties of the roots of Eq. (3.2) can judge the stability and the instability conditions of the given problem.

Before dealing with the dispersion relation (3.2) in detail, we first consider the case when the effect of mass and heat transfer across the interface is negligible (i.e. $\alpha = 0$). In this case, Eq. (3.2) reduces to

$$a_0\omega^2 - b_1\omega + a_{20} = 0. \tag{3.3}$$

It follows that the system is stable, provided that

$$b_1^2 - 4a_0a_{20} \geq 0, \tag{3.4}$$

it follows that

$$g(\rho_2-\rho_1) - \sigma k^2 - kH_1H_2(\mu_2-\mu_1)^2 \times (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2\coth kh_1\coth kh_2) \times (\mu_1\coth kh_1 + \mu_2\coth kh_2)^{-1} + k\rho_1\rho_2(u_1-u_2)^2 \times \coth kh_1\coth kh_2(\rho_1\coth kh_1 + \rho_2\coth kh_2)^{-1} \leq 0. \tag{3.5}$$

As a special case, for two semi-infinite layers and in the absence of magnetic field, the stability condition (3.5) reduces to that obtained earlier by Chandrasekhar [4] (page 485), and therefore his results are recovered. Otherwise, the above condition shows that the surface tension has a stabilizing influence; while the streaming has a destabilizing one. The magnetic field has a stabilizing effect if

$$\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2\coth kh_1\coth kh_2 > 0, \tag{3.6}$$

and vice versa.

In the limit case when $kh_1 \gg 1$ and $kh_2 \gg 1$, i.e. coth $kh_1 \simeq 1$ and coth $kh_2 \simeq 1$ (this is the case of two semi-infinite fluid layers), the condition (3.6) is reduced to [10]

$$\cos(\theta_1+\theta_2) > 0. \tag{3.7}$$

Therefore, for two semi-infinite fluids, the oblique magnetic field has a stabilizing effect if $\cos(\theta_1+\theta_2) > 0$ and a destabilizing effect if $\cos(\theta_1+\theta_2) < 0$. It is dear that when $\cos(\theta_1+\theta_2) = 0$, the magnetic field has no implication on the stability criterion.

Once more, we return to the dispersion relation (3.2) where the parameter α (the coefficient of mass and heat transfer) does not equal zero. We know from the Routh-Hurwitz criterion [23] that necessary and sufficient conditions for stability (in other words, to have the imaginary part of ω less than zero) are

$$a_1 > 0, \tag{3.8}$$

and $$a_0b_2^2 - a_1b_1b_2 + a_{20}a_1^2 \leq 0. \tag{3.9}$$

Since α is always positive, condition (3.8) is automatically satisfied. While condition (3.9), after some re-arrangement, gives

$$g(\rho_2-\rho_1) - \sigma k^2 - kH_1H_2(\mu_2-\mu_1)^2 \times (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2\coth kh_1\coth kh_2) \times (\mu_1\coth kh_1 + \mu_2\coth kh_2)^{-1} + k\rho_1\rho_2(u_1 - u_2)^2\coth kh_1\coth kh_2(\rho_1\coth kh_1 + \rho_2\coth kh_2)^{-1} \times\left[1 + \frac{(\rho_1-\rho_2)^2\coth kh_1\coth kh_2}{\rho_1\rho_2(\coth kh_1 + \coth kh_2)^2}\right] \leq 0. \tag{3.10}$$

In the case of absence of magnetic fields, $\mu_1 \to \mu_2$, condition (3.10) reduces to that, in a pure hydrodynamical fluid, early obtained by Hsieh [20] and therefore his results are recovered. In this case, the stability condition differs from that of the classical Kelvin-Helmholtz problem by the additional last term. It is somewhat surprising that the parameter α does not appear in this expression. Thus this expression is valid even for infinitesimal α, and yet when $\alpha = 0$, the last term is absent. When α is infinitesimally small, the additional effect on the growth rate of the instability is also infinitesimally small. Note also that the critical wavenumber above which the system will be stable can be determined from this case, only for some physical problems of interest when kh_1 and kh_2 are very small or very large individually or together. The influence of the oblique magnetic field depends strongly on the angles θ_1 and θ_2. The case when $\theta_1 = \theta_2 = 0$, which corresponds to tangential magnetic field, shows a stabilizing effect of the magnetic field. This result coincides with the well known result given by Zelazo and Melcher [7]. Also, the case when $\theta_1 = \theta_2 = \pi/2$ corresponds to normal magnetic field, shown thereby a destabilizing influence. This is in agreement with earlier results obtained by Cowley and Rosensweig [8].

In what follows, numerical illustrations of the stability criterion are made. The calculations will be performed with inequality (3.10). Therefore, we shall introduce some particulars of the system under consideration. The goal is to investigate the influence of some physical quantities on the stability of the system. Because of our aim based on the effect of an oblique magnetic field together with the mass and heat transfer parameters on the stability conditions, the numerical estimations considered, especially, the effect of these parameters. In the following figures, the solid curve refers to the case of absence of mass and heat transfer parameter while the dotted one stands for its presence. Also, the letter U stands for unstable regions while the letter S denotes stable ones.

The case of pure hydrodynamic ($\mu_1 \to \mu_2$) is plotted in Fig. 1, where $|u| = \sqrt{(u_1 - u_2)^2}$ is graphed versus k. The plane is partitioned into stable and unstable parts, which makes the destabilizing influence of streaming obvious. This figure is drawn corresponding to the case of immiscible fluids ($\alpha = 0$). It is found that the critical instability occurs at the point (2.89, 27.37). The effect of the mass and heat transfer parameter, in the absence of magnetic field, as well as the inclination angels θ_1 and θ_2 are displayed in Figs 2, 3 and 4 to indicate the $(|u| - k)$ plane.

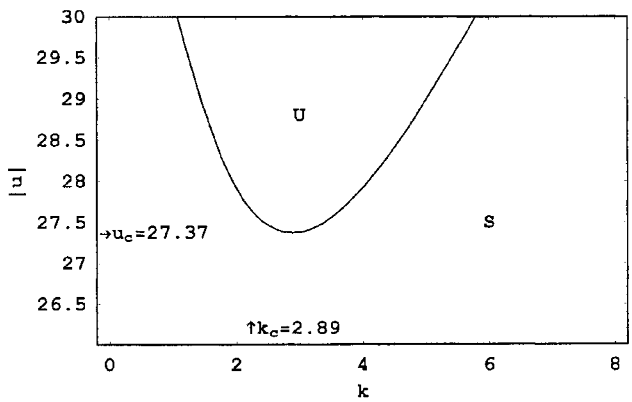

Fig. 1. Stability diagram on the $(|u|-k)$ plane, according to the condition (3.10), for a system having the particulars: $\rho_1 = 0.7\,g/cm^3$, $\rho_2 = 0.38\,g/cm^3$, $h_1 = 0.5\,cm$, $h_2 = 1.0\,cm$, $g = 981\,cm/s^2$, $\sigma = 29\,dyne/cm$ and $H_1 = 0$ in the absence of mass and heat transfer.

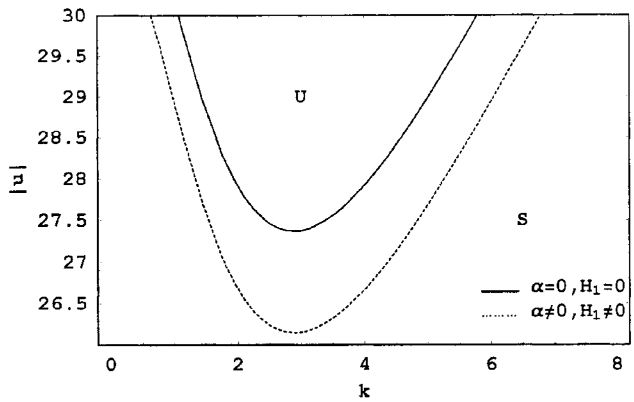

Fig. 2. Stability diagram for the system considered in Fig. 1, but the dotted line corresponds to the presence of mass and heat transfer. The region between the two curves is a newly formed unstable region created as an implication of the effect of mass and heat transfer across the interface.

Figure 2 is plotted to indicate the influence of the parameter α. It is apparent from inspection of this graph that the presence of mass and heat transfer gives a new unstable region bounded between these transition curves. This destabilizing influence is an early important phenomenon discovered by Hsieh [20], in plane geometry, Nayak and Chakraborty [22], in cylindrical geometry, and by several other researchers for inviscid flow through the linear stability theory [23]. The presence of a uniform oblique magnetic field is pictured in Figs 3 and 4 in the $(|u| - k)$ plane. In Fig. 3, where $0 < \theta_1 + \theta_2 < \frac{\pi}{2}$, a newly formed unstable region is obtained due to the presence of this oblique field. The oblique magnetic field, here, behaves like the normal one. The same sense is observed in Fig. 4 where $\frac{\pi}{2} < \theta_1 + \theta_2 < \frac{3\pi}{2}$ [10]. A comparison between these curves shows that the instability enhances as $\frac{\pi}{2} < \theta_1 + \theta_2 < \frac{3\pi}{2}$ especially at small values of the wave number k. The results of calculations are displayed in Figs 5, 6, 7 and 8 to indicate the $(\log H_1^2 - k)$ plane. The special case of a tangential magnetic fIeld is plotted in Fig. 5. It is apparent

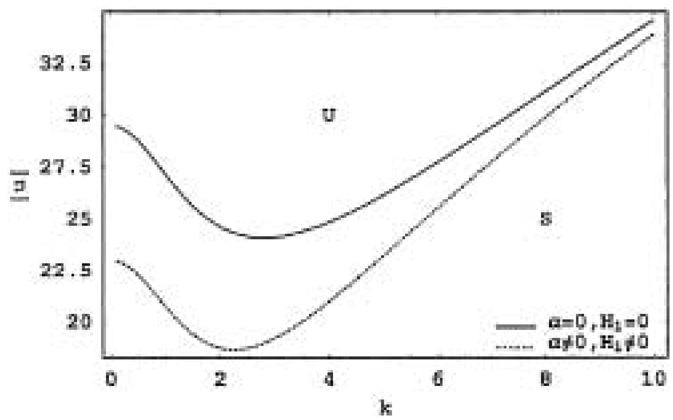

Fig. 3. Stability diagram for the system considered in Fig. 2 but with $\theta_1 = 25°$, $\theta_2 = 54.44°$, $\mu_1 = 5.1\,\mathrm{H/m}$, $\mu_2 = 1.7\,\mathrm{H/m}$, $H_1 = 10\,\mathrm{A/m}$ and $h_1 = 0.1\,\mathrm{cm}$.

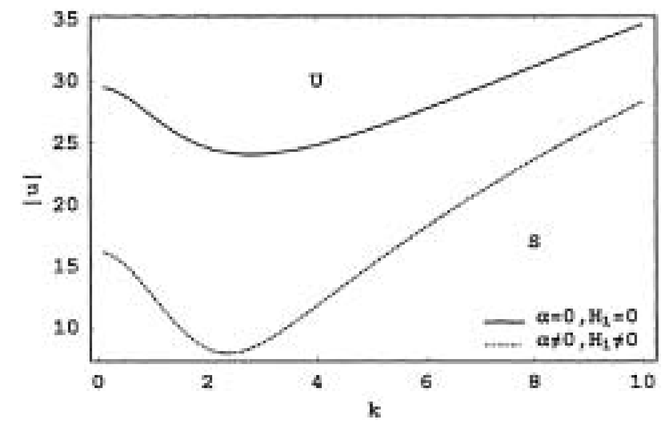

Fig. 4. Stability diagram for the system considered in Fig. 3 but with $\theta_1 = 35°$, $\theta_2 = 64.54°$.

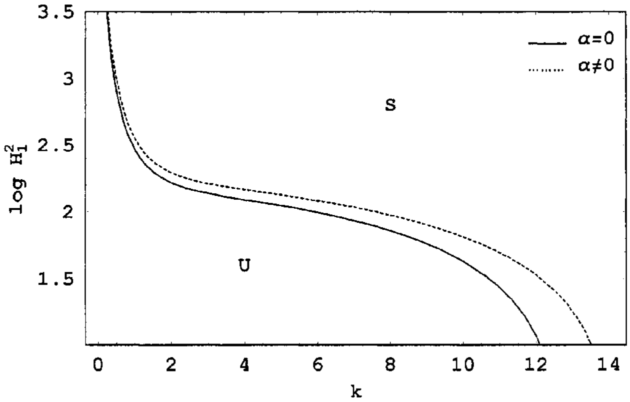

Fig. 5. Stability diagram on the $(\log H_1^2 - k)$ plane according to (3.10), for the system considered in Fig. 3 but with $u_1 = 27\,\mathrm{cm/s}$, $u_2 = -13\,\mathrm{cm/s}$ and $\theta_1 = \theta_2 = 0$.

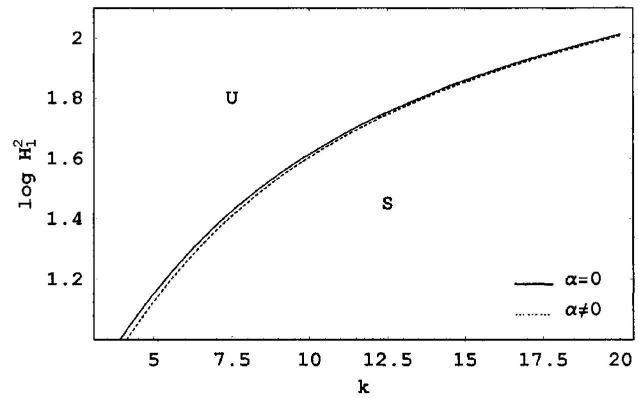

Fig. 6. As in Fig. 5 except that $u_1 = 10\,\mathrm{cm/s}$, $u_2 = -5\,\mathrm{cm/s}$ and $\theta_1 = \theta_2 = \frac{\pi}{2}$.

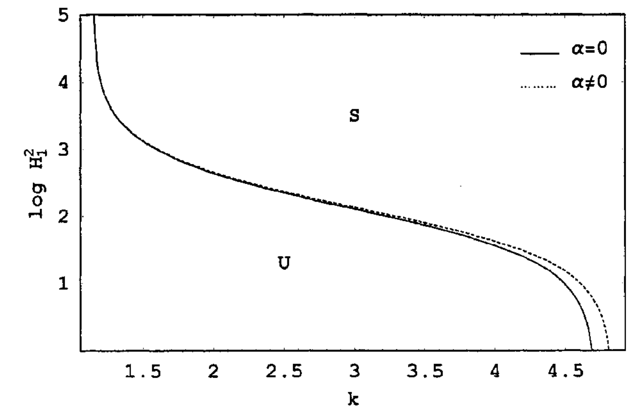

Fig. 7. As in Fig. 6 except that $\theta_1 = 8°$ and $\theta_2 = 22.9°$.

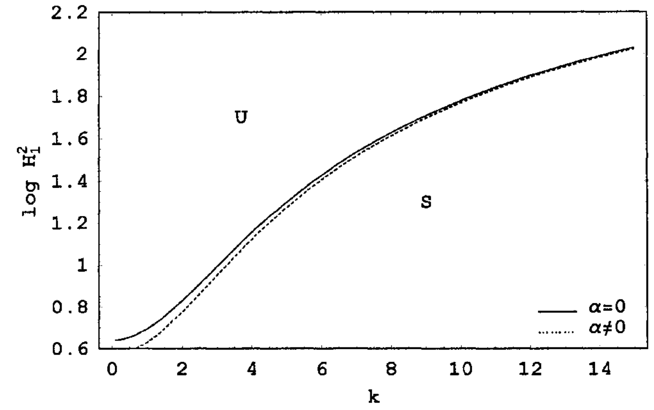

Fig. 8. As in Fig. 6 except that $\theta_1 = 60°$ and $\theta_2 = 79.1°$.

the well known results of the destabilizing influence of the parameter α, specially at large values of the wave number. Figure 6 considers the case of a normal magnetic field. It is clear that the normal field is strictly destabilizing. The presence of an oblique magnetic field is pictured in Figs 7 and 8. In Fig. 7, when $0 < \theta_1 + \theta_2 < \frac{\pi}{2}$, the oblique field behaves like the tangential one. The case when $\frac{\pi}{2} < \theta_1 + \theta_2 < \frac{3\pi}{2}$ is graphed in Fig. 8. In contrast to the previous Fig. 7, the oblique field plays the role of the normal field. Thus the stability condition of the oblique magnetic field depends stongly on the values of θ_1 and θ_2 [10].

4. Hill's equation

In the absence of mass and heat transfer (i.e. $\alpha = 0$), and using the transformation

$$\gamma(t) = f(t)\exp\left[-\int_0^t \left(\frac{ib_1}{2a_0}\right)\mathrm{d}t\right], \tag{4.1}$$

Eq. (2.29) reduces to

$$\frac{\mathrm{d}^2 f}{\mathrm{d}t^2} + [\delta + Q(t)]f = 0, \tag{4.2}$$

where

$$\delta = \left[\frac{k(\rho_1 \coth kh_1 + \rho_2 \coth kh_2)[g(\rho_1 - \rho_2) + \sigma k^2] - k^2 \rho_1 \rho_2 (u_1 - u_2)^2 \coth kh_1 \coth kh_2}{(\rho_1 \coth kh_1 + \rho_2 \coth kh_2)^2}\right], \tag{4.3}$$

and $\quad Q(t) = \beta H_1(t) H_2(t), \qquad (4.4)$

$$\beta = \frac{k^2(\mu_2 - \mu_1)^2(\cos\theta_1 \cos\theta_2 - \coth kh_1 \coth kh_2 \sin\theta_1 \sin\theta_2)}{(\mu_1 \coth kh_1 + \mu_2 \coth kh_2)(\rho_1 \coth kh_1 + \rho_2 \coth kh_2)}.$$

Equation (4.2) is the well-known Hill's differential equation. The nature of the solution of this differential equation governs the fluctuations of the amplitude of the disturbed interface, and it will therefore determine the parametric excitation of magnetic surface waves.

We determine the values of δ for which the solutions of Hill's Eq. (4.2) are stable. Following the method of Magnus and Winkler [25], one can show that for Hill's Eq. (4.2) there exist two monotonically increasing infinite sequences of real numbers

$$\delta_0, \delta_1, \delta_2, \ldots \tag{4.5}$$

and

$$\delta_1', \delta_2', \delta_3', \delta_4', \ldots \tag{4.6}$$

such that Eq. (4.2) has a solution of period π if and only if

$$\delta = \delta_n, \qquad n = 0, 1, 2, \ldots \tag{4.7}$$

and a solution of period 2π if and only if

$$\delta = \delta_n', \qquad n = 0, 1, 2, 3, \ldots \tag{4.8}$$

δ_n and δ_n' satisfy the inequalities

$$\delta_0 < \delta_1' \le \delta_2' < \delta_1 \le \delta_2 < \delta_3' \le \delta_4' < \delta_3 \le \delta_4 < \cdots, \tag{4.9}$$

and the relations

$$\lim_{n\to\infty} \delta_n^{-1} = 0, \qquad \lim_{n\to\infty} (\delta_n')^{-1} = 0. \tag{4.10}$$

The solutions of Eqs (4.2) are stable in the intervals

$$(\delta_0, \delta_1'), (\delta_2', \delta_1), (\delta_2, \delta_3'), (\delta_4', \delta_3), \ldots \tag{4.11}$$

At the end points of these intervals the solutions (4.2) are, in general, unstable. This is always true for $\delta = \delta_0$. The solutions of Eq. (4.2) are stable for $\delta = \delta_{2n+1}$ or δ_{2n+2} if $\delta_{2n+1} = \delta_{2n+2}$, and they are stable for $\delta = \delta_{2n+1}'$ or δ_{2n+2}' if $\delta_{2n+1}' = \delta_{2n+2}'$.

For complex values of δ, Eq. (4.2) always has unstable solutions and it cannot happen here (see 4.3).

δ_n are the roots of the equation $\Delta(\delta) = 2$ and δ_n' are those of $\Delta(\delta) = -2$, where

$$\Delta(\delta) = f_1(\pi, \delta) + f_2'(\pi, \delta). \tag{4.12}$$

The intervals of instability $(-\infty, \delta_0)$ will always be present (the zeroth interval of instability) and define the first interval of instability (δ_1', δ_2').

We observe that neither an interval of stability nor an interval of instability can ever shrink to a point. The intervals of stability can never disappear, but two of them can combine to form a single one, $\delta_{2n+1} = \delta_{2n+2}'$ or $\delta_{2n+1}' = \delta_{2n+2}'$. However, the intervals of instability (with the exception of the zeroth interval) may disappear altogether. This takes place if $Q(t)$ is constant (i.e. for the case of constant oblique magnetic field).

A region in the real (δ, Q) plane will be called a region of absolute stability for functions of class P, if, for any point in this region, (4.2) has stable solutions for all functions $Q(t)$ (Eq. (4.4)) where $H_1(t)H_2(t)$ of class one is bounded by the curves

$$\left.\begin{aligned}&\beta_{n+1} = \pm\left[\frac{4(n+1)\delta^{\frac{1}{2}}}{\pi}\right]\cot\left[\frac{\pi\delta^{\frac{1}{2}}}{2(n+1)}\right],\\&n^2 < \delta < (n+1)^2, \quad \beta_n = \pm 2\delta\left(1-\frac{n}{\delta^{\frac{1}{2}}}\right),\\&\delta > 1,\ n \geq 1,\ \delta = 0 \qquad \text{for } n = 0,\end{aligned}\right] \tag{4.13}$$

and is such that none of these curves is contained in its interior.

The open region bounded by these curves is maximal; for any point outside or on the boundary of this region, there exists a bounded function (4.2). Also, let m be a real variable, $0 \leq m \leq 1$, and let

$$M = \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}s}{(1-m^2\sin^2 s)^{\frac{1}{2}}}, \quad N = \int_0^{\frac{\pi}{2}} (1-m^2\sin^2 s)^{\frac{1}{2}}\mathrm{d}s. \tag{4.14}$$

Then the curves defined for $n = 0, 1, 2, \ldots$ by

$$\left.\begin{aligned}\beta_{n+1} &= \pm 8.3^{-\frac{1}{2}}\pi^{-2}(n+1)^2 M\,[M^2(m^2-1)\\&\qquad + 2MN(2-m^2) - 3N^2]^{\frac{1}{2}},\\\delta_{n+1} &= 4\pi^{-2}(n+1)^2[M^2(m^2-1) + 2MN],\\\delta &> 0,\end{aligned}\right] \tag{4.15}$$

bound the region of absolute stability of functions of class two.

The boundary points do not belong to the region since for

$$\begin{aligned}\delta + Q(t) &= 4\pi^{-2}(n+1)^2 M^2(1+m^2)\\&\quad - 8\pi^{-2}(n+1)^2 m^2 M^2 sn^2\left[\frac{2(n+1)Mt}{\pi}\right],\end{aligned} \tag{4.16}$$

the differential Eq. (4.2) has only one periodic solution (and, therefore, at least one unbounded solution).

The periodic solution (with period π or 2π) is

$$f_p = sn\,\tau, \qquad \tau = \frac{2(n+1)Mt}{\pi}, \tag{4.17}$$

where $sn\,\tau$ is the Jacobian elliptic function with module m and period $4M$.

Also, for the function of class ∞, the region of absolute stability is bounded by the curves

$$\begin{aligned}&(\delta_{n+1} + \beta_{n+1})^{\frac{1}{2}}\tan\left[\frac{\pi(\delta_{n+1}+\beta_{n+1})^{\frac{1}{2}}}{4(n+1)}\right]\\&\quad = (\delta_{n+1} - \beta_{n+1})^{\frac{1}{2}}\cot\left[\frac{\pi(\delta_{n+1}-\beta_{n+1})^{\frac{1}{2}}}{4(n+1)}\right],\end{aligned} \tag{4.18}$$

where $n = 0, 1, 2, \ldots$, and where the region does not contain any of these curves in its interior. If one of the square roots should be imaginary, the functions tan and cot have to be replaced by the corresponding hyperbolic functions.

Also, if a and b are real numbers and

$$a^2 \leq \delta + Q(t) \leq b^2, \tag{4.19}$$

then the solution of Eq. (4.2) will be stable for all possible $\delta + Q(t)$ satisfying this condition if and only if the interval (a^2, b^2) does not contain the square of an integer.

In the following subsection, we shall consider Mathieu's equation as special.

4.1. *Stability analysis of Mathieu equation*

If we take $Q(t) = -2q\cos 2t$, where $Q(t+\pi) = Q(t)$. Equation (4.2) is then reduced to

$$\frac{\mathrm{d}^2 f}{\mathrm{d}t^2} + (\delta - 2q\cos 2t)f = 0, \tag{4.20}$$

where $q = -\beta/2$.

Equation (4.20) is well known as the canonical form of Mathieu's equation which is a linear differential equation with periodic coefficients. Equations similar to this equation appear in many problems in applied mathematics such as stability of a transverse column subjected to a periodic longitudinal load, stability of periodic solutions of a non-linear conservative system, electromagnetic wave propagation in a medium with periodic structure, and the excitation of certain electrical systems. The solutions of the Mathieu equation can be, under certain conditions, periodic where the system becomes stable. The condition for the periodic Mathieu functions depends on the relation between the parameters δ and q. The $(\delta-q)$-plane is divided into stable and unstable regions bounded by the characteristic curves of Mathieu functions. The general solution of Eq. (4.20) is stable if the point (δ, q) in the $(\delta-q)$-plane lies in a stable region, otherwise it is unstable.

According to Floquet's theorem [26], the general periodic solution of the Mathieu differential equation given by (4.20) can be written as

$$f_P = F_1 e^{\varrho t}\Re(t) + F_2 e^{-\varrho t}\Re(t), \tag{4.21}$$

where $\Re(t)$ is a periodic function in t of period π or 2π; F_1, F_2 are arbitrary constants and ϱ is a parameter given by the following relation:

$$\sin^2(i\pi\varrho) = \Delta(0)\sin^2\left(\frac{1}{2}\pi\delta^{\frac{1}{2}}\right), \tag{4.22}$$

where $\Delta(0)$ is an infinite Hill's determinat, depending on ϱ and q and taking the form

$$\Delta(0) \simeq 1 - \frac{\pi\delta^2\cot\left(\frac{\pi\delta^2}{2}\right)}{4\delta^{1/2}(\delta-1)} \tag{4.23}$$

It is seen from Eq. (4.21) that if ϱ is purely imaginary, the solution for f_p will be bound as $t \to \infty$ and the system is stable. The characteristic curves of the Mathieu functions and the regions of stability and instability are discussed by MacLachlan [27]. In the $(\delta-q)$-plane, the regions in which the values of δ and q yield imaginary values of ϱ are stable regions. On the other hand, if ϱ is real, the solution for f_p will tend to ∞ as $t \to \infty$. The unstable regions in the

$(\delta - q)$-plane are the regions in which the values of δ and q correspond to real values of ϱ. The boundary curves of these regions are symmetric about the δ-axis. On the other hand, we assume that q is small (which is a good approximation to high frequency fields or large wavenumbers). Following Morse and Feshbach [28], one can show that the solution of Eq. (4.21) will be bounded as $t \to \infty$ provided that q and δ satisfy the following inequality:

$$q^2 - 4\delta q + 2\delta(1-\delta) \geq 0. \tag{4.24}$$

Also, if $\delta + Q(t) \geq 0$, and

$$\int_0^{\pi} (\delta + Q(t))\mathrm{d}t < \frac{64}{3\pi^2}\left[\int_0^{\pi/2} \frac{\mathrm{d}s}{(1+\sin^2 s)^{1/2}}\right]^4 = \frac{1}{12}\left[\frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})}\right]^4, \tag{4.25}$$

then the solutions of Mathieu's Eq. (4.20) are stable.

5. Damped Mathieu equation

5.1. *Stability behaviour due to an oscillatory magnetic field*

To examine the parametric excitation of surface waves, we shall consider the following special case

$$H_1(t) = H_0 \cos \Omega t, \tag{5.1}$$

where H_0 is the field amplitude and Ω its frequency.

In this case, Eq. (2.29) becomes

$$a_0 \frac{\mathrm{d}^2\gamma(t)}{\mathrm{d}t^2} + (a_1 + \mathrm{i}b_1)\frac{\mathrm{d}\gamma(t)}{\mathrm{d}t} + (a_2^* + a_3\cos^2\Omega t + \mathrm{i}b_2)\gamma(t) = 0, \tag{5.2}$$

where a_0, a_1, b_1 and b_2 are given by Eqs (2.30–2.34), while

$$a_2^* = k[g(\rho_1 - \rho_2) + k^2\sigma - k(\rho_1 u_1^2 \coth kh_1 + \rho_2 u_2^2 \coth kh_2)],$$

$$a_3 = \mu^* H_0^2, \quad \text{and}$$

$$\mu^* = \frac{k^2(\mu_1 - \mu_2)^2(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2\coth kh_1\coth kh_2)\mu_1\sin\theta_1}{\mu_2\sin\theta_2(\mu_1\coth kh_1 + \mu_2\coth kh_2)}$$

Equation (5.2) represents the Mathieu equation with damped terms. Such an equation has growth rate solutions and the stability analysis is rather complex. To economize this complexity, we shall construct the stability configuration through a marginal state analysis. Thus, we shall be dealing with the periodic solutions for this equation. To accomplish this marginal state, two conditions must be satisfied: The necessary and sufficient conditions of stability are, respectively,

$$a_0 \frac{\mathrm{d}^2\gamma(t)}{\mathrm{d}t^2} + \mathrm{i}b_1\frac{\mathrm{d}\gamma(t)}{\mathrm{d}t} + (a_2^* + a_3\cos^2\Omega t)\gamma(t) = 0, \tag{5.3}$$

$$\text{and} \quad a_1\frac{\mathrm{d}\gamma(t)}{\mathrm{d}t} + \mathrm{i}b_2\gamma(t) = 0, \quad a_1 \neq 0. \tag{5.4}$$

It is worthwhile to observe that for non zero a_1, the equation that governs the marginal state can be formulated by combining the necessary condition (5.3) with the sufficient condition (5.4) into a single condition. This can be accomplished by eliminating the damping term $\frac{\mathrm{d}\gamma(t)}{\mathrm{d}t}$ between them. Thus, one gets

$$\frac{\mathrm{d}^2\gamma(\tau)}{\mathrm{d}\tau^2} + (\kappa - 2\lambda\cos 2\tau)\gamma(\tau) = 0, \tag{5.5}$$

where $\tau = \Omega t$,

$$\text{and} \quad \kappa = \frac{1}{\Omega^2 a_0}\left(a_2^* + \frac{b_1 b_2}{a_1}\right) + \frac{\mu^*}{2a_0\Omega^2}H_0^2$$

$$\text{and} \quad \lambda = -\frac{\mu^*}{4a_0\Omega^2}H_0^2.$$

As given in the previous section, inequality (4.24), the stability criterion reduces the problem of the bounded region of the Mathieu functions. In terms of the magnetic field H_0^2, the above condition can be arranged in the form

$$(H_0^2 - H_1^*)(H_0^2 - H_2^*) \geq 0 \tag{5.6}$$

where H_1^* and H_2^* are the transition curves which separate stable from unstable regions. They are given by

$$H_{1,2}^* = \frac{8}{\mu^*}\left(P - \Omega^2 \pm \sqrt{(P-\Omega^2)\left(\frac{3}{2}P - \Omega^2\right)}\right). \tag{5.7}$$

The inequality (5.6) can be satisfied as

$$H_0^2 > H_1^* \quad \text{or} \quad H_0^2 < H_2^*; \quad H_1^* > H_2^*. \tag{5.8}$$

The goal in what follows is to determine the numerical profiles of the stability pictures in the case of an oscillating oblique magnetic field. In fact, the following calculations include only the stability analysis near the marginal state. These calculations are displayed in Figs 9–13. Our attention is focused on the influence of the field frequency Ω. The stability criterion (5.6) is displayed in the $(\log H_0^2 - k)$ plane. In these figures, the plane is partitioned by the transition curves H_1^* and H_2^* into stable and unstable parts. Figure 9 is displayed for the case when $\Omega = 5\,\text{Hz}$. According to Floquet theory [27], the region bounded by the two branches of the transition curves H_1^* and H_2^* is unstable, while the area outside these curves is stable. Figures 10 and 11 are displayed to indicate the effect of the frequency Ω in the case of two finite layers. Therefore, Fig. 10 is plotted when $0 < \theta_1 + \theta_2 < \frac{\pi}{2}$ while Fig. 11 is graphed for

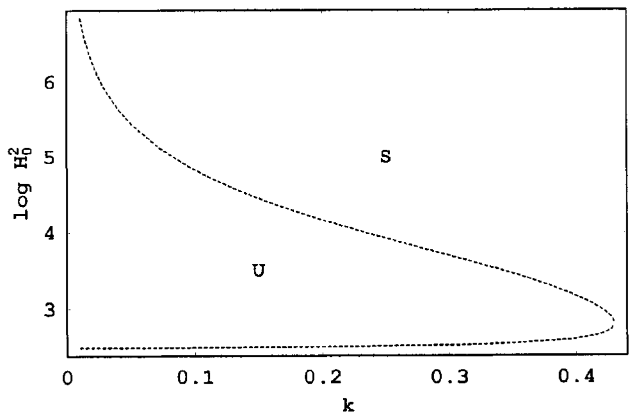

Fig. 9. As in Fig. 6 except that $\theta_1 = 20°$, $\theta_2 = 47.5°$, $h_2 = 0.1\,\text{cm}$, $u_1 = 10\,\text{cm/s}$, $u_2 = 5\,\text{cm/s}$ and $\Omega = 5\,\text{Hz}$.

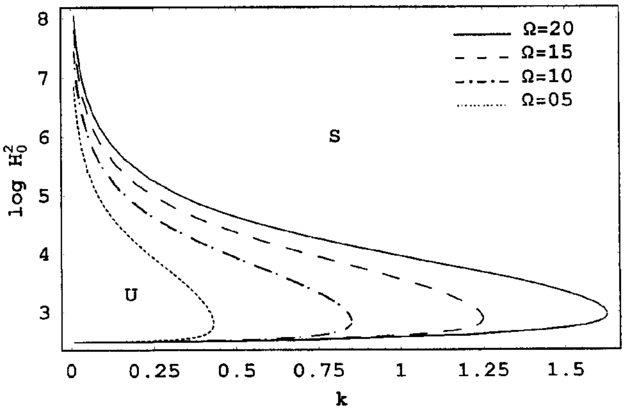

Fig. 10. As in Fig. 9 but for various values of Ω.

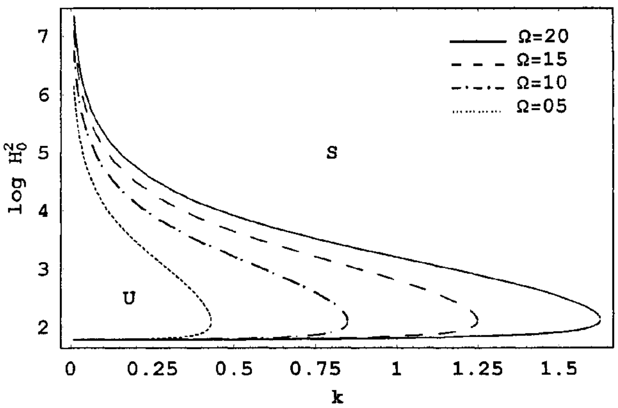

Fig. 11. As in Fig. 10 except that $\theta_1 = 50^\circ$ and $\theta_2 = 74.34^\circ$.

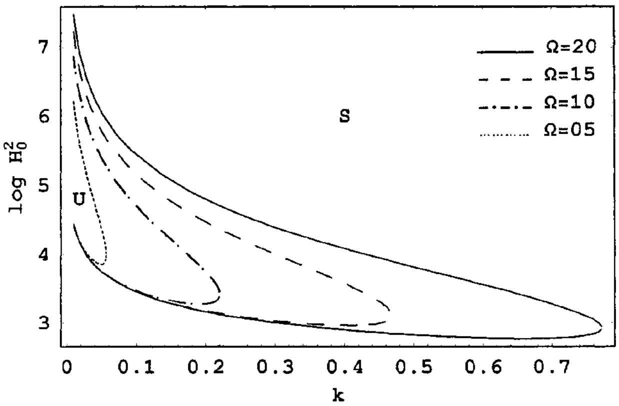

Fig. 12. As in Fig. 10 but for two semi-infinite layers.

$\frac{\pi}{2} < \theta_1 + \theta_2 < \frac{3\pi}{2}$. The figures are pictured for various values of the frequency. In these figures, it is readily seen that as the frequency increases, the width of the unstable region bounded by $(H_1^* - H_2^*)$ increases. This shows the destabilizing influence of the field frequency for all wave numbers k. Thus the frequency of the periodic oblique magnetic field has a destabilizing role regardless of whether the value of $\theta_1 + \theta_2$ is greater or less than $\frac{\pi}{2}$. Figures 12 and 13 are drawn in the special case of two semi-inifinte layers. Figure 12 is computed for the case when $\frac{\pi}{2} < \theta_1 + \theta_2 < \frac{3\pi}{2}$. The figure includes the transition curves H_1^* and H_2^*. The same

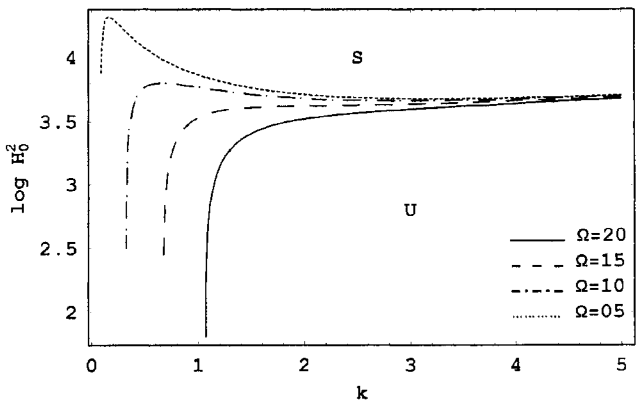

Fig. 13. As in Fig. 12 but for $\theta_1 = 20^\circ$ and $\theta_2 = 47.5^\circ$

influence of the parameter Ω, as gained by Figs 10 and 11, is obtained. The case when $0 < \theta_1 + \theta_2 < \frac{\pi}{2}$ is displayed in Fig. 13. The numerical calculations in this figure show that $H_2^* < 0$. Thus the transition curve $H_0^2 = H_2^*$ has no implication on the stability criteria. It follows that the stability condition reduces to $H_0^2 > H_1^*$. It is apparent that as the parameter Ω increases the stable region increases, which shows stabilizing influence of the field frequency specially at small values of the wave number k. Therefore, for two semi-infinite layers, the frequency of the oblique magnetic field plays a dual role in the stability criteria. This role depends on if the value of $\theta_1 + \theta_2$ is greater or less than $\frac{\pi}{2}$.

5.2. *The Rayleigh-Taylor instability*

We shall examine the parametric excitation of surface waves, for simplicity, in the case of Rayleigh-Taylor instability. According to Floquet theory [26], the parameter space is partitioned into stable and unstable regions. In order to determine the transition curves that separate stable from unstable solutions of the characteristic Eq. (5.2), we shall make use of a small dimensionless parameter ε, which is defined by

$$H_0 = \sqrt{\varepsilon}\,\hat{H}, \tag{5.12}$$

where $\hat{H}$ is associated with the magnitude of the magnetic field intensity.

In the case of Rayleigh-Taylor instability, Eq. (5.2) reduces to

$$\frac{d^2\gamma(\tau)}{d\tau^2} + 2A\frac{d\gamma(\tau)}{d\tau} + (B + R\varepsilon\cos 2\tau)\gamma(\tau) = 0, \tag{5.13}$$

where the coefficients A, B and R are given by

$$A = \frac{\alpha(\coth kh_1 + \coth kh_2)}{2\Omega(\rho_1\coth kh_1 + \rho_2\coth kh_2)},$$

$$B = \frac{k^2}{\Omega^2(\rho_1\coth kh_1 + \rho_2\coth kh_2)}\left(\frac{g(\rho_1 - \rho_2)}{k} + k\sigma\right) + \varepsilon R,$$

$$R = \frac{\mu^*\hat{H}^2}{2\Omega^2(\rho_1\coth kh_1 + \rho_2\coth kh_2)}.$$

To study the stability of Eq. (5.13), using Floquet theory, it can be shown that this equation possesses a solution of the form

$$\gamma(\tau) = \chi(\tau)\exp(\nu\tau), \tag{5.14}$$

where ν is, in general, a complex constant depending on all parameters of the system and is called the characteristic exponent. $\chi(\tau)$ is a periodic function having the same period as the time dependent coefficients in the equations. When the real part of ν is negative the response decays; when the real part of ν is zero, the response is finite and bounded; and when the real part of ν is positive, the response grows. Thus the values of the parameters for which the real part of ν is zero divide the parameter space into regions of stability and instability.

The transition curves separating stability from instability may be obtained by making use of Whittaker's technique [29]. The details are given in a previous work [30]. Away from details, one gets

$$B = A^2 - \frac{R^2\varepsilon^2}{8} + O(\varepsilon^3), \tag{5.15}$$

$$B = 1 + A^2 \pm \frac{R\varepsilon}{2} - \frac{R^2\varepsilon^2}{32} + O(\varepsilon^3), \tag{5.16}$$

$$B = 4 + A^2 + O(\varepsilon^3). \tag{5.17}$$

In what follows, we shall give numerical calculations of the system under consideration by drawing transition curves. The transition curves are represented by Eqs (5.15), (5.16) and (5.17) in the $(B-\varepsilon)$-plane. The values of B, as described by these equations are the critical values of the disturbances. These critical values, which are known as transition curves, separate the stable from unstable regions. From Floquet theory [26], the region bounded by the two branches of the transition curves is unstable while the area outside these curves is a stable region. In fact, the stability discussions of the parametric curves in the figures are restricted by the condition

$$B = \frac{1}{\Omega^2(\rho_1 \coth kh_1 + \rho_2 \coth kh_2)} \times \left[k(g(\rho_1-\rho_2)) + k^2\sigma + \frac{\varepsilon}{2}\mu^*\hat{H}^2\right], \tag{5.18}$$

which may be represented in the (B$-\varepsilon$) plane, by a straight line with a slope depending on the sign of μ^*. This sign depends, in fact, on the thicknesses of the two layers as well as the inclination of the oblique magnetic field. In the following figures, B is plotted versus k, thus condition (5.18) is represented by a curve in the figures. The intersection of the condition (5.18) with the characteristic curves partitions these curves into stable and unstable parts. In these figures, the transition curves are represented by solid curves while the condition (5.18) is represented by a dotted one. The intersection of this dotted curve with the unstable regions occurs at the resonance modes. These resonance modes appear due to the periodicity of the oblique periodic magnetic field.

The results of the following calculations are displayed in Figs 14–16 to indicate the effect of the mass and heat transfer parameter (α) in the $(B-k)$ plane. Figure 14 includes all the transition curves and the condition (5.18), which is a curve intersecting the unstable regions at which resonance modes occur. In this fig., we consider the case of immiscible fluid $\alpha = 0$. Figures 15 and 16 are computed from the same system but when $\alpha = 3$ and $7\,\text{gm/cm}^3\text{s}$ respectively. A comparison between these figures shows the destabilizing influence of the mass and heat transfer parameter.

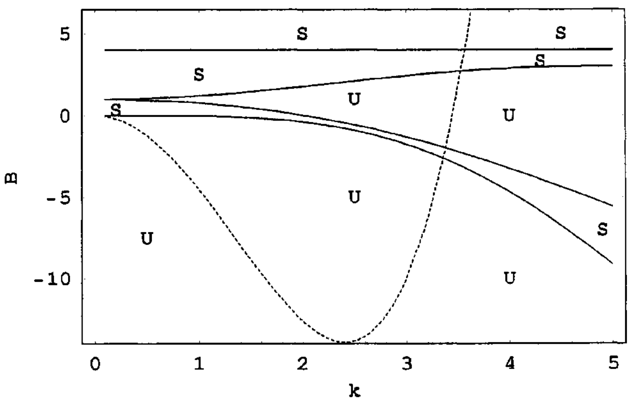

Fig. 14. Stability diagram on the $(B-k)$ plane, for the system considered in Fig. 9 but with $\theta_1 = 30°$, $\theta_2 = 60°$, $\rho_1 = 0.38\,\text{g/cm}^3$, $\rho_2 = 0.7\,\text{g/cm}^3$, $u_1 = u_2 = 0$, $H_1 = 10\,\text{A/m}$ and $\alpha = 0.01\,\text{g/cm}^3\text{s}$.

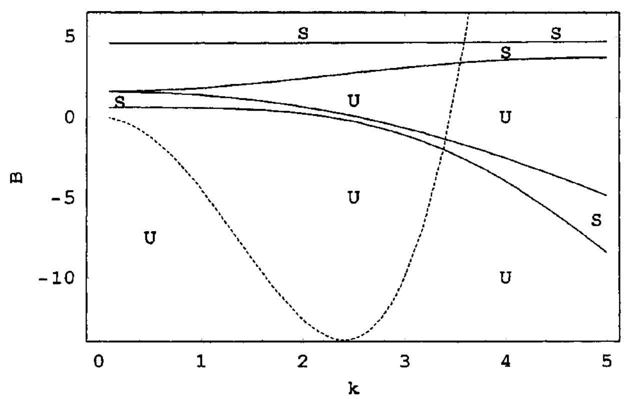

Fig. 15. As in Fig. 14 except that $\alpha = 3\,\text{g/cm}^3\text{s}$.

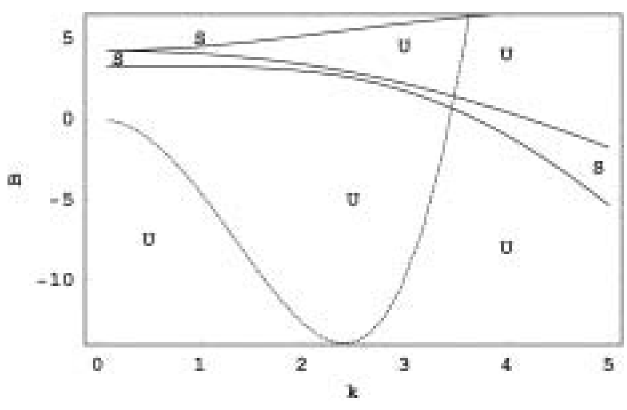

Fig. 16. As in Fig. 14 except that $\alpha = 5\,\text{g/cm}^3\text{s}$.

6. Conclusion

The ferrohydrodynamic flow on a horizontal interface between two inviscid fluids has been studied. The two fluids are enclosed between two horizontal rigid plates in parallel with the interface. The interface admits the presence of mass and heat transfer. The system is acted upon by a time-dependent oblique magnetic field. No free currents at the surface of separation is assumed. In the stationary state, the fluids are uniformly streaming parallel to each other. The stability analysis is based on a linear perturbation theory. Through this analysis, a second-order differential equation with variable coefficients is obtained. The stability of the system is analytically discussed and the results are numerically confirmed. The conclusions may be drawn in two categories as follows:

6.1. *The case of a uniform oblique field*

(1) The tangential field has a stabilizing influence, while the normal one has a destabilizing role.
(2) The effect of the oblique field, in the case of two semi-infinite layer, depends on the values of θ_1 and θ_2.
(3) The mass and heat transfer has no effect on the Rayleigh-Taylor problem.
(4) For the Kelvin-Helmholtz problem, the stability criterion is independent of mass and heat transfer parameter, but is different from that in the same problem without mass and heat transfer.
(5) The mass and heat transfer has a destabilizing influence on the Kelvin-Helmholtz model.

6.2. *The case of a time-dependent field*

(1) Hill's equation is obtained for two immiscible fluids and the stability intervals are obtained by means of Magnus and Winkler [25].
(2) The parametric excitation of the oscillatory oblique field results in a damped Mathieu's equation.
(3) The field frequency has a destabilizing influence regardless on the angle of inclination of the field.
(4) In the case of two semi-infinite layers, the frequency plays a dual role on the stability criteria depending on if the value of $\theta_1 + \theta_2$ is greater or less than $\frac{\pi}{2}$.

Finally, it is found that the mass and heat transfer parameter has a destabilizing influence regardless on the mechanism of the field.

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